Exponents for the Tails of Distributions in some Polling Models *

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Abstract

The tail asymptotics of the distribution of the waiting-time W in some polling models is investigated. When this is of the form $\mathbf{P}[W > x] \sim \alpha x^{\beta} e^{-\eta x}$ for some α, β, η, we show how to calculate the exponents β and η, and we establish the extent and form of their dependence on the distributions of the service-time and switchover-time. The exponents are expressed in terms of the fixed points and Lyapunov exponents of a dynamical system which we associate with the recursion which is used to calculate the moment generating functions of the waiting time.

Keywords: Polling Systems, Tail Asymptotics, Recursion, Lyapunov Exponents, Dynamical Systems

1 Introduction

This paper is motivated by recent work of Choudhury and Whitt [2] in which numerical transform inversion was used to calculate exponents for the tail distribution of various quantities of interest in a class of polling models. As those authors point out, whereas the relationships between transforms of (the distributions of) these quantities have been used to obtain their low order moments, far less is known about their distributions. For models with Poisson arrivals, general service-time and switchover-time distributions, and various service disciplines, [2] establishes numerical consistency of the tail asymptotics of, for example, the steady state waiting time W, with the form

$$\mathbf{P}[W > x] \sim \alpha x^{\beta} e^{-\eta x}. \tag{1}$$

Numerical values for the prefactor α and the exponents β and η are obtained for given model parameters.

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Here we use analytic methods to explore the relationship between the exponents β and η, and their dependence (and sometimes independence) on the service-time and switchover-time distributions. As in [2], the main analytic tool is the recursion relation among the transforms, proposed by Eisenberg [3] as a route to obtaining steady state transforms; see also Konheim and Meister [6] for another early paper, and the book of Tagaki [8] for a more recent review. Roughly speaking, this recursion specifies a dynamical system in transform space, and the exponents β and η are expressible in terms of the fixed points and Lyapunov exponents (see e.g. [5]) of this dynamical system. These exponents are the only ones compatible with the asymptotics (1). Our results are contingent on stability properties of the dynamical system which we are able to establish only for $N = 2$ sources. However, our formulae for β and η give values which are in agreement with those obtained numerically in the examples for $N = 3$ which were treated in [2]. The question of whether the asymptotics are necessarily of the form (1) is left open; for an example of a queueing model where they do not hold see [4].

The structure of the relationship between the exponents β and η is detailed in Theorems 1, 2 and 3 below, along with the method for their calculation. We can summarize the relationship as follows. Suppose that the tail of the distribution function R of the switchover-time is characterized by a exponents β_R, η_R in the sense that for some α_R, $1 - R(x) \sim \alpha_R x^{\beta_R} e^{-x\eta_R}$. It seems reasonable to expect that the dominant effect on the tail of the waiting-time distribution will be from whichever of the distributions of the service-time and the switchover-time has the heavier tail. This turns out to be the case: when the queueing system is stable,

$$\eta = f(\min[\eta_R, \eta']), \tag{2}$$

where η' depends only on the arrival rate and the service-time distribution and f is an increasing function which we will specify. So, as expected, the smaller of the exponents η_R, η' determines the dominant exponential asymptotic behavior of the waiting-time distribution. When the switchover time distribution has the heavier tail, i.e., $\eta_R < \eta'$, then $\beta = \beta_R$ (except in some exceptional circumstances). If $\eta \geq \eta'$, then β depends on both distributions in a more complicated manner.

The paper is organized as follows. In section 2 we recall the relation between the exponents β, η of the tail distribution of a random variable, and the singular behavior of its exponential moment generating function. In section 3 we recast the recursive scheme of [2] for the calculation of Laplace transforms as a dynamical system, whose properties are investigated in section 4. The main theorems of the paper are proved in section 5, in which the exponents β and η are expressed in terms of the fixed points of the dynamical system and their associated Lyapunov exponents. Finally, in section 6 we test our formulae for the exponents against those calculated in the numerical studies of [2].

We finish the introduction by noting that the techniques of this paper should be more widely applicable: we analyzed the asymptotics of the waiting-time, but in principle one could analyze any variable whose exponential moment generating function is expressed recursively. The substitution of other service disciplines would not alter the principle, nor would the consideration of joint distributions of random variables from different queues in the system.

2 Exponents and Singularities

We will work mostly with exponential moments of various quantities of interest. Let us start by relating the form of their singularities to asymptotics of the form (1).

Lemma 1 *Let X be a non-negative random variable and suppose that for $\beta > -1$*

$$\mathbf{P}[X > x] \sim \alpha x^{\beta} e^{-\eta x} \;, \quad (x \to \infty). \tag{3}$$

Then

$$\lim_{\eta'' \nearrow \eta'} \frac{\log \mathbf{E}[e^{\eta'' X}]}{-\log(\eta' - \eta'')} = \beta_{\eta'} + 1, \tag{4}$$

where

$$\beta_{\eta'} = \begin{cases} -1 & \text{if } \eta' < \eta \\ \beta & \text{if } \eta' = \eta \\ \infty & \text{if } \eta' > \eta \end{cases} \tag{5}$$

Our strategy in this paper will be to establish (4) (when X is the stationary waiting time W) for some finite η' and $\beta_{\eta'} > 1$ which we will determine. Then (5) means that these are the only exponents consistent with asymptotics of the form (1).

Proof of Lemma 1: Let U denote the distribution of X. From (1) it follows that for y sufficiently large

$$\int_y^x (1 - U(x'))e^{\eta x'} dx' \sim c x^{\beta+1}, \quad (x \to \infty). \tag{6}$$

(Here, and below, $c, c'c''$ are constants). But integrating by parts we have

$$\int_y^x (1 - U(x'))e^{\eta x'} dx' = \eta^{-1}(1 - U(x'))e^{\eta x'}|_{x'=y}^{x} + \eta^{-1} \int_y^x dU(x')e^{\eta x'} dx', \tag{7}$$

whence

$$U_\eta(x) := \int_0^x dU(x')e^{\eta x'} \sim c' x^{\beta+1} \quad (x \to \infty). \tag{8}$$

Thus by Karamata's Tauberian Theorem (see Thm. 1.7.1 of [1])

$$\begin{aligned} \hat{U}_\eta(t) &= \int_0^\infty dU_\eta(x')e^{-tx'} = \int_0^\infty dU(x')e^{(\eta - t)x'} & (9) \\ &= \mathbf{E}[e^{(\eta - t)X}] \sim c'' t^{-(\beta+1)} \quad (t \to 0^+), & (10) \end{aligned}$$

from which the result follows. ■

3 The Models and their Iterations

Consider a cyclic symmetric polling model of N queues, each with Poisson arrivals of rate λ, service-time distribution B, non-zero switchover-time distribution R. We shall work with the gated service discipline: when the server arrives at a queue it serves only those customers present at the time of arrival. However, we expect our technique to be applicable to other service disciplines.

Let $\tilde{B}$ denote the exponential moment generating function of the service-time distribution, i.e., $\tilde{B}(s) = \int_0^\infty dB(x)e^{sx}$; and similarly for R. We will make the following assumptions concerning the distributions of the specified and the derived quantities occurring here; these will be in force for the remainder of the paper.

Hypothesis 1 *(i)* $d := \sup\{y \in \mathbf{R} : \tilde{B}(y) < \infty\}$ *is strictly positive*

(ii) For some $\alpha_R, \eta_R > 0$ *and* $\beta_R > -1$

$$1 - R(x) \sim \alpha_R x^{\beta_R} e^{-\eta_R x}. \tag{11}$$

(iii) Stability: the offered load $N\lambda\tilde{B}'(0) < 1$.

(iv) Non-triviality: $N\lambda(\tilde{B}(s) - 1) > s$ *for some* $s > 0$.

Note that (iv) is satisfied, for example, if $\tilde{B}(s) \to \infty$ as $s \nearrow d$. Let $\tilde{W}$ denote the exponential moment generating function of the stationary waiting time W, i.e., $\tilde{W}(s) = \mathbf{E}[e^{sW}]$. We shall rephrase some results from [2, 8] to show the following, the proof of which is deferred to the end of this section.

Proposition 1

(i)

$$\tilde{W}(s) = \frac{(1-N\rho)}{Nr} \frac{\tilde{p}(1+s/\lambda) - \tilde{p}(\tilde{B}(s))}{\lambda\left(1 + s/\lambda - \tilde{B}(s)\right)}, \tag{12}$$

where r *is the mean switchover time,* $\rho = \lambda\tilde{B}'(0)$, *and* $\tilde{p}$ *is the moment generating function of the stationary queue length* Q *i.e.* $\tilde{p}(z) = \mathbf{E}[z^Q]$.

(ii)

$$\tilde{p}(\tilde{B}(s)) = \prod_{k=1}^{\infty} \tilde{R}(y^{(k-1)}) \tag{13}$$

where

$$y^{(n)} = \sum_{k=1}^{N} D(y^{(n-k)}) \qquad (n \geq 0) \quad \text{with} \tag{14}$$

$$D(y) = \lambda(\tilde{B}(y) - 1) \tag{15}$$

and with initial conditions

$$y^{(-1)} = s, \quad y^{(-2)} = y^{(-3)} = \cdots y^{(-N)} = 0. \tag{16}$$

On the basis of (1) and Lemma 1 we desire to determine the location and nature of the singularity in $\tilde{W}$ on the positive half-axis. This occurs at the singularity of $s \mapsto \tilde{p}(1 + s/\lambda)$: since from [8] $\tilde{p}(1+s/\lambda)$ (being the exponential moment generating function of the stationary cycle time between the ends of successive visits to the same queue) is greater than $\tilde{p}(\tilde{B}(s))$ (the exponential moment generating function of stationary time spent by the server at a given queue). As we shall see, a necessary condition for (13) to be finite is that $\lim_{n\to\infty} y^{(n)} = 0$. We will study the properties sequence $(y^{(n)})_{n\geq -N}$ within the following framework. Define $M : [0,d)^N \to \mathbf{R}^N$ by

$$M(y_1,\ldots y_n) = \left(y_2,\ldots y_N, \sum_{i=1}^{N} D(y_i)\right). \tag{17}$$

Note that D is invertible on $[0,d)$. Let $\mathbf{e}$ denote the vector $(0,\ldots,0,1) \in \mathbf{R}^N$. Set $\mathbf{R}(y_1,\ldots y_N) = \tilde{R}(y_N)$. Then for $s \geq 0$ it follows that

$$\tilde{p}(\tilde{B}(s)) = H(s), \quad \text{where} \quad H(x) = \mathbf{H}(x\mathbf{e}) \quad \text{and} \quad \mathbf{H}(\mathbf{x}) = \prod_{k=1}^{\infty} \mathbf{R}\left(M^k\mathbf{x}\right). \tag{18}$$

Expressing (13) in terms of (18), $\tilde{p}(1+s/\lambda) = H(D^{-1}(s))$. Combining with (12) the behavior of $\tilde{W}$ near its singularity on the positive half-axis is

$$\tilde{W}(s) \sim \text{const. } H(D^{-1}(s)). \tag{19}$$

Proof of Proposition 1:

(i) follows from the corresponding statement on p. 110 of [8] for the Laplace transforms $\hat{B}$ and $\hat{W}$ of the service-time and waiting-time distributions.

To get (ii) we start from the iteration of Proposition 2.3 of [2] for Laplace transforms (see (2.14)-(2.16) there) which we adapt to the symmetric case and re-express in terms of moment generating functions as follows:

$$\tilde{p}(\tilde{B}(s)) = \prod_{i=1}^{\infty} \tilde{R}(y^{(k-1)}) \tag{20}$$

with

$$y^{(k)} = y^{(k-1)} + \lambda\left(z^{(k)}_{w(-k)} - z^{(k-1)}_{w(-k)}\right), \tag{21}$$

the $\mathbf{z}^{(k)}$ are an N-tuples $(z^{(k)}_1,\ldots z^{(k)}_N)$ defined recursively by

$$z^{(k)}_j = \begin{cases} z^{(k-1)}_j, & j \neq w(-k) \\ \tilde{B}(y^{(k-1)}), & j = w(-k) \end{cases}, \tag{22}$$

where $w(j) = j \bmod N$, $\mathbf{z}^{(0)} = (\tilde{B}(s), 1, \ldots, 1)$ and

$$y^{(0)} = \lambda(\tilde{B}(s) - 1) = D(s). \tag{23}$$

Now (22) can be iterated back to give

$$z^{(k-1)}_{w(-k)} = \begin{cases} z^{(k-2)}_{w(-k)} = \ldots = z^{(k-N)}_{w(-k)} & = \tilde{B}(y^{(k-N-1)}), \quad k > N \\ z^{(k-2)}_{w(-k)} = \ldots = z^{(0)}_{w(-k)} & = \begin{cases} \tilde{B}(s), & k = N \\ 1, & k < N \end{cases} \end{cases} \tag{24}$$

Imposing the initial conditions (16), we see that *for all* $k > 0$,

$$z_{w(-k)}^{(k-1)} = \tilde{B}(y^{(k-N-1)}). \tag{25}$$

Now it remains to establish (14) by induction from $n = 0$: there $y^{(0)} = D(s) = \sum_{i=1}^{N} D(y^{(-i)})$ by (16) and (23), as required. For the induction step, note that (14) for $n = k - 1$, (21) and (25) together imply (14) for $n = k$. ■

4 The Dynamical System Generated by M

Consider the discrete dynamical system obtained by iterating M. First, we must fix up what happens at the upper boundary of $[0, d)^N$ since $D(d) = \infty$. So we extend $[0, d)^N$ by adjoining a point, denoted by ∞, and extend M (and hence $\mathbf{H}$) to $\mathcal{D} = [0, d]^N \cup \{\infty\}$ by setting $M\mathbf{y} = \infty$ on $[0, d]^N \setminus [0, d)^N$ and $M\infty = \infty$. Define the usual partial order $\preceq$ on $[0, d]^N$ by $\mathbf{x} \preceq \mathbf{y}$ iff $x_i \leq y_i$ for all $1 \leq i \leq N$ and extend to $\mathcal{D}$ by $\mathbf{x} \preceq \infty$ for all $\mathbf{x} \in \mathcal{D}$. Similarly, write $\mathbf{x} \prec \mathbf{y}$ if $x_i < y_i$ for all $1 \leq i \leq N$.

Now for $x > 0$, we expect (and shall prove) that the existence of $H(x)$ requires that $M^n(x\mathbf{e}) \to \mathbf{0} = (0, \ldots, 0)$ as $n \to \infty$. Let us denote the attractor set A_0 of $\mathbf{0}$ under (the dynamics generated by) M by

$$A_0 = \{\mathbf{x} \in \mathcal{D} : \lim_{n\to\infty} M^n\mathbf{x} = \mathbf{0}\}. \tag{26}$$

Define the least upper bound $\hat{\mathbf{y}} = \hat{y}\mathbf{e}$ of those elements in A_0 which are proportional to $\mathbf{e}$, i.e.,

$$\hat{y} = \sup\{y \geq 0 : y\mathbf{e} \in A_0\}. \tag{27}$$

Furthermore, for each $\mathbf{y} \in \mathcal{D}$ define the supremum of the projections onto $\mathbf{e}$ of all points along the trajectory generated by M starting from $\mathbf{x}$: i.e.

$$\mathbf{F}(\mathbf{x}) := \sup_{k\geq 0}(M^k\mathbf{x})_N, \tag{28}$$

and set

$$F(y) = \mathbf{F}(y\mathbf{e}). \tag{29}$$

Proposition 2 *(i) Let* $\mathbf{F}(\mathbf{y}) < \eta_R$. *Then* $\mathbf{H}(\mathbf{y})$ *is finite when* $\mathbf{y} \in A_0$ *and hence* $\tilde{p}(\tilde{B}(y))$ *is finite for* $y \in [0, \hat{y})$.

(ii) $\mathbf{H}(\mathbf{y})$ *is infinite when* $\mathbf{y} \in \mathcal{D} \setminus A_0$ *and hence* $\tilde{p}(\tilde{B}(y))$ *infinite for* $y \geq \hat{y}$.

The proof of this and subsequent results of the paper depends on the following properties of M which we shall first establish.

Proposition 3 *(i). M is continuous on $[0, d)^N$.*

(ii). M preserves the partial order $\preceq$ on $\mathcal{D}$ i.e. $\mathbf{x} \preceq \mathbf{y}$ implies $M\mathbf{x} \preceq M\mathbf{y}$.

(iii). M and its iterates are convex on $\mathcal{D}$ under the order $\preceq$ i.e. for all $\nu \in [0,1]$ and $k \in \mathbf{Z}_+$, $M^k(\nu\mathbf{x} + (1-\nu)\mathbf{y}) \preceq \nu M^k\mathbf{x} + (1-\nu)M^k\mathbf{y}$.

(iv). Let $0 \preceq \mathbf{y} - \mathbf{x}$. Then

$$\mathbf{0} \preceq \nabla M(\mathbf{x}) \cdot (\mathbf{y} - \mathbf{x}) \preceq M\mathbf{y} - M\mathbf{x} \preceq \nabla M(\mathbf{y}) \cdot (\mathbf{y} - \mathbf{x}). \tag{30}$$

(v). M has exactly three fixed points: ∞, $\mathbf{0} = (0, \ldots, 0)$ and $\mathbf{y}^* = (y^*, \ldots, y^*)$ where y^* is the unique positive solution of $y^* = ND(y^*)$.

(vi). A_0 is a convex set.

(vii). $\mathbf{x}, \mathbf{y} \in \mathcal{D}$, $\mathbf{y} \in A_0$ and $\mathbf{x} \preceq \mathbf{y}$ together imply that $\mathbf{x} \in A_0$.

(viii). A_0 is open in $\mathcal{D}$.

(ix). $\mathbf{y}^*$ lies on the boundary $\partial A_0 = \bar{A}_0 \setminus A_0$.

(x). A_0 is strictly larger than $\{\mathbf{0}\}$.

(xi). ∂A_0 is invariant under M i.e. $M\partial A_0 \subseteq \partial A_0$.

Proof:

(i). This follows since $\tilde{B}$ is the exponential moment generating function of a non-negative random variable and is hence continuous on the interior $(-\infty, d)$ of its domain.

(ii). Likewise, $\tilde{B}$, and hence D, is increasing. Thus M preserves $\preceq$.

(iii). D is convex by Hölder's inequality so that M is convex under the order $\preceq$. Iterates of M are then convex by (ii).

(iv). This is an immediate consequence of (iii) and the positivity of D'.

(v). From the definition of M, at a fixed point $\mathbf{x}$ then for all i, $x_i = \sum_{j=1}^N D(x_j) = ND(x_i)$. $\mathbf{0}$ and ∞ are clearly two fixed points. The existence of exactly one more, namely $\mathbf{y}^*$, follows from the convexity of D, and that by Hypothesis 1(iii) $ND'(0) < 1$ and so by Hypothesis 1(iv) the equation $y = ND(y)$ has a solution which is finite and strictly positive.

(vi). Suppose $M^k\mathbf{x}$ and $M^k\mathbf{y}$ converge to $\mathbf{0}$ as $k \to \infty$. Since M^k is convex so does $M^k(\nu\mathbf{x} + (1-\nu)\mathbf{y}) \preceq \nu M^k\mathbf{x} + (1-\nu)M^k\mathbf{y}$ for all $\nu \in [0,1]$.

(vii). This follows from (vi) and the fact that $\mathbf{0} \in A_0$.

(viii). Let $\mathbf{y} \in A_0 \setminus \{\mathbf{0}\}$. By definition $M^k\mathbf{y}$ converges to $\mathbf{0}$ as $k \to \infty$ so that for all n there exists $m > n$ such that $M^m\mathbf{y} \prec M^n\mathbf{y}$. Hence there is a neighborhood of $M^m\mathbf{y}$ all of whose members are $\prec M^n\mathbf{y}$ and is hence by (vii) a subset of A_0. So by (i)—the continuity of M—there is a neighborhood of $\mathbf{y}$ which is mapped into A_0 by M^n and which is hence itself a subset of A_0.

(ix).

$$\nabla M(\mathbf{y}) = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \\ D'(y_1) & D'(y_2) & D'(y_3) & \dots & D'(y_N) \end{pmatrix}. \tag{31}$$

$\nabla M(\mathbf{y}^*)$ is got by replacing all elements of the last row by $g := D'(y^*) > 0$. By the Perron-Frobenious theorem, $\nabla M(\mathbf{y}^*)$ has a unique eigenvalue μ of maximum modulus which is strictly positive and for which the corresponding eigenvector $\mathbf{v}$ has strictly positive entries. In fact $\mu > 1$. For the characteristic polynomial of $\nabla M(\mathbf{y}^*)$ is

$$C(\mu) = \mu^N - g(1 + \mu + \dots \mu^{N-1}), \tag{32}$$

and so $C(\mu) \to +\infty$ as $\mu \to \infty$. But since D is convex with $D(0) = 0$ and $ND(y^*) = y^*$ we have that $1 < ND'(y^*) = Ng$. Thus $C(1) = 1 - Ng < 0$ and we conclude from continuity that C has at least one root $\mu > 1$. Thus for all $\varepsilon > 0$ sufficiently small $M(\mathbf{y}^* - \varepsilon\mathbf{v}) \prec \mathbf{y}^* - \varepsilon\mathbf{v}$. Hence the sequence $(M^k(\mathbf{y}^* - \varepsilon\mathbf{v}))_{k \in \mathbf{Z}_+}$ is non-increasing in $\preceq$ and so converges to a limit which is a fixed point for M. Since this fixed point is $\prec \mathbf{y}^*$ it must be $\mathbf{0}$. Thus we have demonstrated the existence of members of A_0 which are arbitrarily close to $\mathbf{y}^*$ and can conclude that $\mathbf{y}^* \in \partial A_0$.

(x). This follows from (ii), (viii) and (ix), since $\mathbf{y}^* \neq \mathbf{0}$.

(xi). Let $\mathbf{y} \in \partial A_0$ and let $(\mathbf{y}_\alpha)$ be an sequence in A_0 converging to $\mathbf{y}$. Then $(M\mathbf{y}_\alpha)$ is a sequence in A_0 converging to $M\mathbf{y}$ which is hence in $\bar{A}_0$. But $M\mathbf{y} \notin A_0$ (otherwise $\mathbf{y} \in A_0$, which is not the case since A_0 is open) and hence $M\mathbf{y} \in \partial A_0$.

■

Proof of Proposition 2:

(i): By Prop. 3(ix),(x), $\mathbf{0}$ attracts every point in some open ball containing it. Furthermore, from (31), $\nabla M(\mathbf{0})$ takes the same form as $\nabla M(\mathbf{y}^*)$ except with g replaced by $g' = D'(0) < 1/N$. Likewise, if μ is an eigenvalue of $\nabla M(\mathbf{0})$, then $\mu^N - g'(1 + \mu + \dots + \mu^{N-1}) = 0$. So if μ were real and not strictly less than 1, we would have

$$N < \frac{1}{g'} = \mu^{-1} + \mu^{-2} + \dots + \mu^{-N} \leq N. \tag{33}$$

a contradiction. This applies in particular to the Perron-Frobenious eigenvalue of $\nabla M(\mathbf{0})$, and so $\|\nabla M(\mathbf{0})\| < 1$. Combining this with the attracting property of $\mathbf{0}$, then for every point $\mathbf{y} \in A_0$, $M^k\mathbf{y}$ converges exponentially to $\mathbf{0}$ for k large enough i.e. there exists $k_0, b, c > 0$ (depending on $\mathbf{y}$) such that $\|M^k\mathbf{y}\| < ce^{-bk}$ for $k > k_0$. The summability of the series $\log \mathbf{H}(\mathbf{y})$ then follows from the fact that $\tilde{R}(0) = 1$ and that $\tilde{R}$ is convex and differentiable on a neighborhood of 0. The second statement then follows from (18).

(ii): Now suppose $\mathbf{y} \in \mathcal{D} \setminus A_0$. Then $\mathbf{y} \succeq \mathbf{z}$ for some $\mathbf{z} \in \partial A_0$, and so by Prop. 3(x),(xi), $\mathbf{R}(M^k\mathbf{y}) > \mathbf{R}(M^k\mathbf{z}) > 1$ and is bounded away from 1 for all k. Hence $\mathbf{H}(\mathbf{y}) = \infty$. The second statement then follows from (18). ■

5 Lyapunov Exponents

Having established the properties of the dynamical system generated by M, we can move on to the main results of the paper. In what follows we shall assume that $\mathbf{y}^*$ attracts $\hat{\mathbf{y}}$ under the dynamics generated by M. Actually, this is something which we shall prove later in the case $N = 2$ (see Proposition 4).

Theorem 1 *Let $F(\hat{y}) < \eta_R$ and assume that $\mathbf{y}^*$ attracts $\hat{\mathbf{y}}$ under the dynamics generated by M. Then the only exponents η and β compatible with (1) are*

$$\eta = D(\hat{y}) \ , \qquad \beta = \frac{\log \tilde{R}(y^*)}{\log \mu} - 1, \tag{34}$$

where μ is the eigenvalue of $\nabla M(\mathbf{y}^*)$ of largest modulus.

Remarks: The condition $F(\hat{y}) < \eta_R$ means that $\mathbf{R}(M^k \hat{\mathbf{y}})$ is not already infinite for some k^{th} point along the trajectory generated from $\hat{\mathbf{y}}$ by M. Since D and $\tilde{R}$ are strictly increasing, so is F, which hence has an inverse. In the notation of (2), $F(\hat{y}) = \eta'$ and $f = D \circ F^{-1}$.

Before embarking on the proof of Theorem 1, we account for the above formula for β by the following heuristic argument. Since by assumption $\mathbf{y}^*$ attracts $\hat{\mathbf{y}}$, then for a $\mathbf{x}$ which is both $\prec$ and close to $\hat{\mathbf{y}}$, the trajectory $M^k\mathbf{x}$ eventually goes to $\mathbf{0}$, but not before having spent a 'middle-portion' of is trajectory close to $\mathbf{y}^*$. Each point in this middle portion makes a contribution of roughly $\tilde{R}(y^*)$ to the product (13). The rate (per iteration) of divergence of the trajectories starting from $\mathbf{x}$ and $\hat{\mathbf{y}}$ is the Lyapunov exponent $\log \mu := \lim_{n\to\infty} n^{-1} \log \|\nabla M^n(\hat{\mathbf{y}}) \cdot (\hat{\mathbf{y}} - \mathbf{x})\|$. Thus for $\mathbf{x} = \hat{\mathbf{y}} - \mu^{-n}\mathbf{e}$ there are roughly n such contributions from the middle portion of the trajectory, and so

$$\frac{\log \mathbf{H}(\mathbf{x})}{-\log(\hat{y} - x)} \approx \frac{\log \tilde{R}(y^*)}{\log \mu}. \tag{35}$$

Proof of Theorem 1: By (1), (4), (18), Lemma 1, and Prop. 2(i), it suffices to establish that

$$\lim_{n\to\infty} \frac{\log H(x_n)}{-\log(\hat{y} - x_n)} = \frac{\log \tilde{R}(y^*)}{\log \mu}. \tag{36}$$

along some sequence $x_n \nearrow \hat{y}$.

Lower Bound: Let $K > 0$ set $x_n = \hat{y} - e^{-Kn}$ and let $\mathbf{x}_n = x_n\mathbf{e}$. Then

$$\frac{\log \mathbf{H}(\mathbf{x}_n)}{-\log(\hat{y} - x_n)} \geq \frac{1}{Kn}\sum_{k=1}^{n} \log \mathbf{R}(M^k \mathbf{x}_n) \geq \frac{1}{Kn}\sum_{k=1}^{n} \log \mathbf{R}(M^k \mathbf{x}_k) \tag{37}$$

since $x_n \geq x_m$ for $n \geq m$, and $\mathbf{R}$ and M are non-decreasing. Thus we can conclude that

$$\lim_{n\to\infty} \frac{\log \mathbf{H}(\mathbf{x}_n)}{-\log(\hat{y} - x_n)} \geq \frac{\log \mathbf{R}(\mathbf{y}^*)}{\log \mu} \tag{38}$$

provided that we can show that $\lim_{n\to\infty} M^n \mathbf{x}_n = \mathbf{y}^*$ for all $K > \log\mu$.

But, by iteration of Prop. 3(iv), we have

$$0 \preceq M^n \hat{\mathbf{y}} - M^n \mathbf{x}_n \preceq \nabla M^n(\hat{\mathbf{y}}) \cdot (\hat{\mathbf{y}} - \mathbf{x}_n) \tag{39}$$

and so

$$\|M^n \mathbf{x}_n - M^n \hat{\mathbf{y}}\| \leq e^{-Kn} \|\nabla M^n \hat{\mathbf{y}}\| \, \|\mathbf{e}\|. \tag{40}$$

Since $\mathbf{y}^*$ attracts $\hat{\mathbf{y}}$ under M

$$\lim_{n\to\infty} \frac{1}{n} \log \|\nabla M^n(\hat{\mathbf{y}})\| \leq \lim_{n\to\infty} \frac{1}{n} \sum_{m=0}^{n-1} \log \|\nabla M(M^m \hat{\mathbf{y}})\| = \log\mu, \tag{41}$$

so that provided $K > \log\mu$ we have

$$\lim_{n\to\infty} n^{-1} \log \|M^n \mathbf{x}_n - M^n \hat{\mathbf{y}}\| < 0, \tag{42}$$

which suffices to establish the required bound.

Upper bound: Now take $K \in (0, \log\mu)$, and perform the decomposition

$$\log \mathbf{H}(\mathbf{x}_n) = \sum_{k=1}^{n} \log \mathbf{R}(M^k \mathbf{x}_n) + \log \mathbf{H}(\mathbf{z}_n) \tag{43}$$

where $\mathbf{z}_n = M^n \mathbf{x}_n$. By assumption, $\mathbf{R}(M^m \mathbf{x}_n) < \mathbf{R}(M^m \hat{\mathbf{y}}) \to \mathbf{R}(\mathbf{y}^*)$ as $m \to \infty$. So if we can show that for all fixed $K < \log\mu$, $\log \mathbf{H}(\mathbf{z}_n)$ is bounded above as $n \to \infty$ along some subsequence, we can conclude that

$$\frac{\log \mathbf{H}(\mathbf{x}_n)}{-\log(\hat{y} - x_n)} \geq \frac{1}{Kn} \sum_{k=1}^{n} \log \mathbf{R}(M^k \hat{\mathbf{y}}) + \mathbf{O}(1/n). \tag{44}$$

Taking the limits $n \to \infty$ (so that $M^n \hat{\mathbf{y}} \to \mathbf{y}^*$) then $K \to \log\mu$ we obtain the desired upper bound.

By repeating the exponential convergence argument used in the proof of Prop. 2(i) one sees that $\mathbf{H}$ is continuous on $A_{\mathbf{0}}$. Since $A_{\mathbf{0}}$ is pre-compact, then in order to establish the boundedness of $\log \mathbf{H}(\mathbf{z}_n)$ as $n \to \infty$, it suffices to show that some subsequence $(\mathbf{z}_n)$ remains bounded away from $\partial A_{\mathbf{0}}$ as $n \to \infty$.

Now by Prop. 3(vii),

$$\limsup_{n\to\infty} \mathbf{z}_n \preceq \limsup_{n\to\infty} M^n \hat{\mathbf{y}} = \mathbf{y}^*. \tag{45}$$

and by Prop. 3(vii), $\mathbf{z} \prec \mathbf{y}^*$ implies $\mathbf{z} \in A_{\mathbf{0}}$. Thus, the only possible limit point of $(\mathbf{z}_n)$ in $\partial A_{\mathbf{0}}$ is $\mathbf{y}^*$. We now show that $(\mathbf{z}_n)$ does not converge to $\mathbf{y}^*$ (although it may have $\mathbf{y}^*$ as a limit point), and that consequently some subsequence of $(\mathbf{z}_n)$ remains bounded away from $\mathbf{y}^*$ and hence from $\partial A_{\mathbf{0}}$.

$M^n \hat{\mathbf{y}} - \mathbf{z}_n = M^n \hat{\mathbf{y}} - M^n \mathbf{x}_n \succeq \mathbf{0}$ and hence

$$\begin{aligned} M^n \hat{\mathbf{y}} - \mathbf{z}_n &\succeq \nabla M^n(\mathbf{x}_n) \cdot (\hat{\mathbf{y}} - \mathbf{x}_n) \qquad \text{by Prop. 3(iv)} && (46)\\ &= e^{-Kn} \nabla M^n(\mathbf{x}_n) \cdot \mathbf{e} && (47)\\ &\succeq e^{-Kn} L_n \cdot \mathbf{e} \succeq \mathbf{0} && (48) \end{aligned}$$

where $L_1 = \nabla M(\mathbf{x}_1)$, and for $n > 1$, L_n is defined recursively by $L_n = \nabla M(\mathbf{z}_{n-1}) \cdot L_{n-1}$. Suppose that $\mathbf{z}_n$ converges to $\mathbf{y}^*$ as $n \to \infty$. Then $\nabla M(\mathbf{z}_n)$ converges to $\nabla M(\mathbf{y}^*)$: ∇M is continuous for the same reason as M is—see Prop. 3(i). Hence by (48)

$$\lim_{n\to\infty} \frac{1}{n} \log \|M^n \hat{\mathbf{y}} - \mathbf{z}_n\| \geq -K + \lim_{n\to\infty} \frac{1}{n} \log \|L_n \cdot \mathbf{e}\| = -K + \log \mu > 0. \tag{49}$$

The last equality follows by general methods in the theory of Lyapunov exponents in [7]. To see it more explicitly, note from (31), the continuity of ∇M and the convergence of $\mathbf{z}_n$ to $\mathbf{y}^*$, that for all $\varepsilon \in (0,1)$ there exists n_ε such that $(1-\varepsilon)\nabla M(\mathbf{y}^*)_{ij} < \nabla M(\mathbf{z}_n)_{ij} < (1+\varepsilon)\nabla M(\mathbf{y}^*)_{ij}$ for all $n > n_0$ and $1 \leq i,j \leq N$. (49) then follows because from the form (31) of ∇M, ∇M^N has strictly positive entries and hence $\mathbf{e}$ has a non-zero projection onto the eigenvector of $\nabla M(\mathbf{y}^*)$ with eigenvalue μ.

Finally, since $\|\mathbf{z}_n - \mathbf{y}^*\| \geq \|M^n\hat{\mathbf{y}} - \mathbf{z}_n\| - \|M^n\hat{\mathbf{y}} - \mathbf{y}^*\|$ and $\lim_{n\to\infty} M^n\hat{\mathbf{y}} = \mathbf{y}^*$, then (49) contradicts the convergence of $\mathbf{z}_n \to \mathbf{y}^*$ along the subsequence. ■

Theorem 1 shows that provided $\tilde{R}(\mathbf{F}(\hat{\mathbf{y}}))$ is finite, η is determined by the service time distribution alone with no further dependence on the switchover-time distribution. Suppose, on the other hand, that $\tilde{R}(\mathbf{F}(\hat{\mathbf{y}})) = \infty$ so that $\mathbf{F}(\hat{\mathbf{y}}) \geq \eta_R$. That is to say, $\mathbf{H}(\mathbf{y})$ blows up even on some $\mathbf{y} \in A_0$ since the last component of $M^k\mathbf{y} \geq \eta_R$ for some k.

Set

$$\breve{y} = F^{-1}(\eta_R), \tag{50}$$

with $\breve{\mathbf{y}} = \breve{y}\mathbf{e}$. We consider first the case that $F(\hat{y})$ is strictly larger than η_R.

Theorem 2 *Let $F(\hat{y}) > \eta_R$. Then $\breve{y} < \hat{y}$, $R(M^k\breve{\mathbf{y}}) = \infty$ for k in some finite subset $\mathcal{K}$ of $\mathbf{N}$ and the only values of η and β compatible with (1) are*

$$\eta = D(F^{-1}(\eta_R)) = D(\breve{y}) \quad \text{and} \quad \beta = |\mathcal{K}|(\beta_R + 1) - 1. \tag{51}$$

One would expect the case $|\mathcal{K}| \neq 1$ to be exceptional in the sense that it requires a complicated relation between R and B to hold. In fact, we can exclude it altogether in the case $N = 2$; see Proposition 4 below.

Proof of Theorem 2: From (19) $\eta = D(\breve{y})$ since H has a singularity at $\breve{y}$. Since $F(\hat{y}) > \eta_R$ and F is increasing, $\breve{y} < \hat{y}$ and by continuity of $\tilde{R}$ there must be at least one index k for which $\mathbf{R}(M^k\breve{\mathbf{y}}) = \infty$. But $\breve{\mathbf{y}} \in A_0$ so there can only be finitely many such indices. Pick a particular such index k and let (x_n) be a sequence increasing to $\breve{y}$ and set $\mathbf{x}_n = x_n\mathbf{e}$. Then $(M^k\mathbf{x}_n)_N$ is a sequence increasing to $\eta_R = (M^k\breve{\mathbf{y}})_N$ and so

$$\frac{\log \mathbf{R}(M^k\mathbf{x}_n)}{-\log(\breve{y} - x_n)} = \frac{\log \tilde{R}\left(-\eta_R + (M^k\breve{\mathbf{y}})_N - (M^k\mathbf{x}_n)_N\right)}{-\log(\breve{y} - x_n)}. \tag{52}$$

By Proposition 3(iv)

$$m_{\mathbf{x}_n}(\breve{y} - x_n) \leq (M^k\breve{\mathbf{y}})_N - (M^k\mathbf{x}_n)_N \leq m_{\breve{\mathbf{y}}}(\breve{y} - x_n), \tag{53}$$

where $m_{\mathbf{x}} = \mathbf{e} \cdot \nabla M^k(\mathbf{x}) \cdot \mathbf{e}$. Thus

$$\lim_{n\to\infty} \frac{\log \mathbf{R}(M^k \mathbf{x}_n)}{-\log(\breve{y} - x_n)} = \lim_{t\to 0+} \frac{\tilde{R}(t - \eta_R)}{-\log t}, \tag{54}$$

$$= \beta_R + 1, \tag{55}$$

where the first equality is due to the continuity of $\mathbf{x} \mapsto m_{\mathbf{x}}$, and the second follows from Hypothesis 1 and Lemma 1. Finally,

$$\log \mathbf{H}(\mathbf{x}_n) = \sum_{k\in\mathcal{K}} \log \mathbf{R}(M^k \mathbf{x}_n) + \sum_{k\notin\mathcal{K}} \log \mathbf{R}(M^k \mathbf{x}_n). \tag{56}$$

The terms in the second sum are bounded a go to 0 exponentially fast as $k \to \infty$, uniformly in n. Hence dividing by $\log(\breve{y} - x_n)$ and taking the limit $n \to \infty$, each term in the first sum yields a contribution $\beta_R + 1$, while the second sum goes to 0. The formula for β then follows by Hypotheses 1 and Lemma 1. ∎

In the borderline case that $F(\hat{y}) = \eta_R$, then it is not difficult to see that β is composed of contributions arising in both the previous theorems.

Theorem 3 *Let $F(\hat{y}) = \eta_R$ and assume that $\mathbf{y}^*$ attracts $\hat{\mathbf{y}}$ under M. Then the only values of η and β compatible with (1) are*

$$\eta = D(F^{-1}(\eta_R)) \quad \text{and} \quad \beta = \log \tilde{R}(y^*)/\log \mu + |\mathcal{K}|(\beta_R + 1) - 1. \tag{57}$$

Proposition 4 *Let $N = 2$. Then*

(i) $F(y) = \left(M^2(y\mathbf{e})\right)_2 = D(y) + D(D(y))$.

(ii) $|\mathcal{K}| = 1$.

(iii) $\mathbf{y}^*$ *attracts* $\hat{\mathbf{y}}$ *under the dynamics generated by* M.

Proof: (i) Let $\mathbf{x} = (x, y) \in \bar{A}_0 \cap \mathcal{S}$ where $\mathcal{S} = \{(x, y) \ : \ x < y\}$. In fact we will prove the slightly stronger result that the maximum of the sequence $((M^n\mathbf{x})_2)_{n\geq 1}$ is achieved at $n = 2$. Set

$$(X, Y) = M^2(x, y) = (X, D(X) + D(y)), \text{ with } X = D(x) + D(y). \tag{58}$$

First we show that $x < y$ implies $Y < y$. Suppose instead that $Y > y$. Then we must have $X < x$, for otherwise $M^2\mathbf{x} \succ \mathbf{x}$ and by Prop. 3(ii), $n \mapsto M^{2n}\mathbf{x}$ is an $\prec$-increasing sequence which precludes $\mathbf{x} \in A_0$; and since D is increasing, $X < x$ implies through (58) that $Y < X$ and hence $Y < X < x < y$: a contradiction.

So $Y < y$ and there remain two alternatives: either $X > x$ or $X < x$. If $X > x$ then $Y > X$ since D is increasing. If instead $X < x$, then $M^2\mathbf{x} \prec \mathbf{x}$ and so $n \mapsto M^n\mathbf{x}$ (n even) is a $\prec$-decreasing sequence. Taking these together, we see that the iterates $M^n\mathbf{x}$ ($n > 0$ even) remain in $\mathcal{S}$ (in which case the second component is strictly decreasing) either indefinitely or up to and including some finite index n_0, after which the iterates are $\prec$-decreasing (in which case the second component is also strictly decreasing). Together, this means that $n \mapsto (M^n\mathbf{x})_2$ ($n > 0$ even) is decreasing. To treat the odd-order iterates, then observe that $(M^{n-1}\mathbf{x})_2 < (M^n\mathbf{x})_2$ for $n > 0$ even, since for even $n \leq n_0$, $(M^{n-1}\mathbf{x})_2 = (M^n\mathbf{x})_1 < (M^n\mathbf{x})_2$ (since $M^n\mathbf{x} \in \mathcal{S}$), while for even $n > n_0$, $(M^{n-1}\mathbf{x})_2 = (M^n\mathbf{x})_1 < (M^{n_0}\mathbf{x})_1 < (M^{n_0}\mathbf{x})_2$. The second equality in the statement follows from the definition of M.

(ii) $|\mathcal{K}| = 1$ since the inequalities above are strict.

(iii) The above proof shows that $n \mapsto (M^n\mathbf{y})_2$ ($n > 0$, even) is a decreasing function bounded below and hence converges to a limit. Hence $\lim_{n\to\infty} M^n\hat{\mathbf{y}}$ exists, and by continuity of M the limit is a fixed point of M. But by Prop.3(v) there is only one such point in ∂A_0, namely $\mathbf{y}^*$.

■

We collect together the results of the foregoing theorems for $N = 2$:

Corollary 1 *Let $N = 2$. Then $F(y) = (M^N(y\mathbf{e}))_N$ and the only exponents η and β compatible with (1) are as follows:*

(i) If $F(\hat{y}) < \eta_R$ then $\eta = D(\hat{y})$ and $\beta = \log\tilde{R}(y^*)/\log\mu - 1$.

(ii) If $F(\hat{y}) > \eta_R$ then $\eta = D(F^{-1}(\eta_R))$ and $\beta = \beta_R$.

(iii) If $F(\hat{y}) = \eta_R$ then $\eta = D(F^{-1}(\eta_R))$ and $\beta = \log\tilde{R}(y^*)/\log\mu + \beta_R$

6 Case-wise Verification for $N = 3$

The foremost question arising is to what extent Proposition 4, and hence the above Corollary, generalize to $N > 2$. Numerical investigations of some trajectories in models for $N = 3$ have not contradicted the statement of the proposition, although the monotone property established in the proof of Proposition 4 (i) was not found to hold in all cases.

In this section we investigate the efficacy of the formulae for η and β from Corollary 1 beyond the domain in which the required assumptions have been substantiated. We calculate exponents and compare with values determined by numerical transform inversion in [2] for two cases with $N = 3$. Our method to find $\hat{\mathbf{y}}$ is by successive iteration from initial points $\mathbf{y} = y\mathbf{e}$. On the assumption that $\hat{\mathbf{y}}$ is attracted to $\mathbf{y}^*$ under M, then those iterations which converge to $\mathbf{0}$ give a lower bound for $\hat{\mathbf{y}}$, while those which go to ∞ give an upper bound.

Example 1 This is Case 2 of Table 5 in [2]. Take $N = 3$, $\lambda = 0.27$. Service is Erlang (E_2) with mean 1 and squared coefficient of variation 0.5, giving

$$D(x) = 0.27\left((1 - x/2)^{-2} - 1\right). \tag{59}$$

Switchover times are exponentially distributed with mean 0.1, giving

$$\tilde{R}(x) = (1 + x/10)^{-1} \quad \text{and} \quad \eta_R = 10. \tag{60}$$

The solution of the equation in Prop 3(v) is

$$y^* = (4 - N\lambda - \sqrt{(4 - N\lambda)^2 - 16(1 - N\lambda)})/2 = 0.259326. \tag{61}$$

The bounding method described above establishes $\hat{y}$ to lie between $\hat{y}_- = 0.4436630$ and $\hat{y}_+ = 0.4436631$. $F(\hat{y}_+) = D(\hat{y}_+) + D(D(\hat{y}_+)) + D(D(\hat{y}_+) + D(D(\hat{y}_+))) = 0.305360 < \eta_R$. Thus we expect case (i) of Corollary 1 to apply. This gives an estimate for η of $D(\hat{y}_+) = 0.175878$. One finds $\mu = 1.11047$ and so $\beta = -0.749238$. Both these figures are in agreement with the (extrapolated) values in [2].

Example 2 This is Case 3 of Table 5 in [2] which is as Example 1 except that the switchover times have Gamma distribution with mean 1 and variance 10. This gives $\eta_R = 1/10 < F(\hat{y}_+)$ and we expect case (ii) of Corollary 1 to apply. $\beta = \beta_R = -9/10$: the value from [2] is -0.8894. $\eta = D(F^{-1}(\eta_R)) = 0.0605904$: the value from [2] is 0.060593.

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