# Allocating commodity resources in aggregate traffic networks

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**Abstract**

We examine the relation between cost and quality in networks which carry aggregate traffic. A powerful tool in this is the burstiness or indifference curve associated with a stochastic traffic flow. We relate burstiness to quality and use this relation to explore the quality experienced by aggregated flows under various rules for allocating resources to them. An example is motivated by the Controlled Load service specification. We show how the imposition of costs associated with buffer space and service capacity leads to the notion of a cost-optimal allocation of resources. This defines the cheapest operating point in a network where resources are commodities to be purchased as necessary to satisfy quality requirements. We define a notion of cost-based admission control: a linear admission rule which can be based on declared or measured traffic parameters.

*Keywords:* Pricing, economies of scale, indifference curve, admission control, performance analysis, large deviations, stochastic networks.

## 1 Introduction

It is critical to provide quality of service (QoS) guarantee in high speed networks in a manner which makes efficient use of network resources. Bandwidth and buffer capacity are the two resources we consider in this paper. For traffic flows which can tolerate a certain buffer delay these are substitutable in the sense that an increase in one may be traded off against a decrease in the other while maintaining quality. Methods to tradeoff bandwidth and buffer in resource allocation are developed in Elwalid, Mitra and Wentworth [10], Low and Varaiya [18, 17], and Lo Presti, Zhang, Kurose and Towsley [24].

Much work has been directed to establishing criteria under which flows may be admitted to networks with a certain fixed set of scarce resources. However, the rapid growth in the number of telecommunications service providers means that another model could be used. In this model we regard bandwidth and buffer space as commodities which are available, possibly from a number of suppliers with potentially different prices. Thus the trade-off between bandwidth and buffer allocation has the potential to be achieved whenever resources are allocated to a flow, rather than at the level of provisioning the network. In such a model the constraint on admission to the network becomes cost-based. As before, the users of the network will have QoS requirements. However, the determining factor as to whether a flow is admitted is not whether the resources are available, but whether the user is prepared to pay the cost of the resources needed in order that their QoS requirements are met.

In this paper we consider traffic flows with variable rate but whose statistical properties are stationary. The notion of QoS we use is that of (sufficiently small) buffer overflow probabilities. Our broad aim is to develop a framework for the efficient allocation of the resources of bandwidth and buffer capacity while minimizing the cost required to achieve a required QoS. As we shall presently describe, one of the potential problem that arises when flows sharing resources, is that heterogeneous characteristics and quality requirements can lead to suboptimal use of resources. However, we shall show that this need not be the case. We shall find the circumstances under which it is possible for flows to specify, out of all resource requests compatible with a desired quality, that which guarantees quality in the shared resource. Roughly speaking, this choice ensures that the parameters of the traffic and resource which determine the frequency of buffer overflow are common across all flows. In some sense, the overflow-determining properties of the heterogeneous flows can be rendered homogeneous over subsets of flows. Furthermore, we show how the relative cost of network resources may be chosen in order to encourage resources to be used in the optimal manner. We shall also investigate how the consequences of such choice propagate in a network.

The approach will be stochastic in that we deal with the statistical multiplexing of flows described by stochastic processes. This is motivated by studies of the deterministic properties of individual sample paths. The form of the tradeoff between bandwidth and buffer is directly used to characterize sample paths in [18]. An arrival process $A = (A_{t,t'})_{t \le t' \in \mathbf{R}}$ specifies the amount of work $A_{t,t'}$ generated in intervals $[t, t')$. To $A$ we can associate its convex decreasing **burstiness curve** (or indifference curve) $b_A(c)$. This is defined as the minimal buffer size $b_A(c)$ to prevent overflow when $A$ is served at a deterministic rate $c$:

$$b_A(c) = \sup_{-\infty < t < t' < 0} \{A_{t,t'} - c(t' - t)\}. \tag{1}$$

This characterization can be thought of as a generalization of the $(\sigma, \rho)$ characterization of Cruz [6] if we treat $\sigma$ as a function of $\rho$. In [18], several burst reducing servers whose output process $A'$ is less bursty than their input process $A$, i.e., $b_{A'}(c) \le b_A(c)$, are identified and their behavior in tandem is studied. Based on these results, in [17], the burstiness of different traffic classes and at different servers is traded off in allocation of bandwidth and buffer in a network of burst reducing servers in order to optimize overall network performance. The deterministic fluid flow model in [17, 18] leads to simpler analysis, but is not suitable to study multiplexing with nonzero loss requirement. In this paper we develop a stochastic framework where such issues can be better examined; for a different approach, see, e.g., Kurose [16], Yaron and Sidi [31] and Chang [3].

The framework which gives rise to stochastic versions of QoS and burstiness is as follows. Use $A$ to model a stochastic traffic flow with stationary increments. We will abbreviate the random quantity $A_{-t,0}$ by $A_t$.

Suppose the aggregate $A^n$ of $n$ independent copies of $A$ feeds a buffer which is drained at a constant rate $nc$ for some $c$. Let $Q^n$ denote the random queue length at time zero. Following Borovkov [1], we can express this pathwise in terms of $A^n$ as

$$Q^n = \sup_{t>0}(A_t^n - cnt). \tag{2}$$

Under very general circumstances the tail probability for $Q^n$ obeys the asymptotic

$$\lim_{n\to\infty} n^{-1} \log \mathbf{P}[Q^n > nb] = -I(c, b) \tag{3}$$

where the **shape function** $I$ is determined by the statistical properties of $A$. This suggests using the following **shape function approximation** for tail probabilities:

$$\mathbf{P}[Q^n > nb] \approx e^{-nI(c,b)} \tag{4}$$

Generally, $I(c, b)$ determines the *economies of scale* which are available through statistical multiplexing. For example, for a broad class of short-range dependent flows $I(c, b) \approx \delta b + \nu$, for large $b$, where $\nu$ is positive for flows with the positive correlation property that its increments are associated. Thus, estimates of system capacity based on the approximation $\mathbf{P}[Q^n > nb] \approx e^{-nI(c,b)} \approx e^{-n(\delta b+\nu)}$ will be less conservative than approximations based on the corresponding *effective bandwidth approximation* $P[Q^n > nb] \approx e^{-n\delta b}$ for such flows. Thus by use of (3) we can aim to develop connection admission control (CAC) criteria which can make better use of system resources than those based on the effective bandwidth approximation; e.g. see Kelly [13], and Gibbens and Hunt [11].

The relation (3) expresses the asymptotic behavior of $Q^n$ as $n$ increases. In view of (2) it is not surprising then that the asymptotic properties of $Q^n$ for large $n$ can be expressed in terms of those of $A^n$. Define the moment generating function (MGF) of an individual flow by

$$\mu(t, \theta) = \log \mathbf{E}[e^{\theta A_t}]. \tag{5}$$

Recall the definition of the Legendre transform $f^*$ of a real function $f$, i.e,

$$f^*(x) = \sup_{\theta\in\mathbf{R}} \{x\theta - f(\theta)\}. \tag{6}$$

Fix some $t > 0$. Then the usual Chernoff bound for sums of independent random variables shows that

$$\mathbf{P}[A_t^n \geq n(ct + b)] \leq e^{-n\mu^*(t,ct+b)}. \tag{7}$$

Duffield [8] has established very general conditions under which that (3) holds with

$$I(c, b) = \inf_{t\geq 0} \mu^*(t, b + ct) = \inf_{t>0} \sup_{\theta\in\mathbf{R}} \{(b + ct)\theta - \mu(t, \theta)\}, \tag{8}$$

where $\mu^*$ denotes the Legendre transform of $\mu$ w.r.t its second argument. It is worth remarking that the conditions established in [8] allow for long range dependent traffic flows; see also Botvich and Duffield

[2], Courcoubetis and Weber [5] and Simonian and Guibert [26] for results for short-range dependent traffic flows.

Actually the results from [8] are not limited to aggregates of identical flows, nor of independent ones. More generally one can consider a sequence of aggregations $A^{(n)}$, with MGF $\mu^{(n)}$. (An i.i.d. aggregation would have $\mu^{(n)} = n\mu$ where $\mu$ is the single flow MGF). For $C = cn$ and $B = bn$ the shape function approximation (4) then becomes $\mathbf{P}[Q^n > B] \approx e^{-I^{(n)}(C,B)}$ where $I^{(n)}$ is derived from $\mu^{(n)}$ using (8). We will use this approximation for aggregate flows in what follows. We will refer to a specific value of the shape function as a **quality**.

The stochastic counterpart of the burstiness curve derives from the shape function. Informally at least, we can define the inverse of $I$ in its second component as $b(c, \varepsilon) = I(c, \cdot)^{-1}(\varepsilon)$. For a homogeneous aggregation we can interpret $b(c, \varepsilon)$ as the buffer level $b$ per flow required in order that the quality, as specified by the log-tail probability $\log \mathbf{P}[Q^n > nb]$, is $\varepsilon$ per flow when the allocated bandwidth is $c$ per flow. Thus $b(c, \varepsilon)$ is the stochastic counterpart of the indifference or burstiness curve (1) in that $\{(c, b) \mid b \geq b(c, \varepsilon)\}$ is the locus of points for which the quality $I(c, b)$ is at least $\varepsilon$. Similarly, $B^{(n)}(C, \varepsilon) = I^{(n)}(C, \cdot)^{-1}(B)$ approximates the buffer required for tail probability $e^{-\varepsilon}$ for the aggregate served at rate $C$.

In Section 2 we will give the formal definition of the burstiness curve in terms of the shape function, and establish its basic properties. In Section 3 we show that the burstiness of a flow is non-increasing on passage through a FIFO queue. This can be seen as a stochastic development of the results from [17] on deterministic flows, and also of asymptotic results for departure processes in the effective bandwidth formalism; see de Veciana, Courcoubetis and Walrand [27] and O'Connell [23]. Large deviation analysis of departure processes in the many sources asymptotic has been performed by Wischik [29], although the focus there was on the aggregate of (independent) components from the outputs of different queues, rather than a single queue as we consider here.

As well as understanding the propagation of burstiness through the network, we also want to be able to express the burstiness of an aggregate of flows in terms of the burstiness of its constituents. This is important in understanding how to allocate resources for the aggregate flows. In Sections 4–6 we analyze the consequences for QoS of two rules for allocating bandwidth and buffer to aggregations:

**Sum rate, sum buffer (SRSB)** The aggregate is allocated the sum over constituents of the bandwidth and buffer capacity. This is motivated in part by the rules for summing Traffic Specifications (TSpec's) when flows are to be combined in the Controlled Load service specification [30]. In Section 5 we investigate the conditions under which the quality of the aggregate, as determined by buffer overflow probabilities, can be at least as great as the minimum quality of its constituent flows.

**Sum rate, maximum buffer (SRMB**) A more stringent allocation of resources to the aggregate is to sum the bandwidths but, in distinction with SRSB, use the maximum buffer allocation of the constituents. In Section 6 we show that flows whose autocorrelation satisfies a certain positivity condition can be (homogeneously) aggregated in this manner without loss of quality.

Since the burstiness curves and shape functions can be regarded as inverses, it is not surprising that our results on combinations of shape functions can also be recast as results about combinations of burstiness curves. We shall obtain stochastic counterparts of the combination law for burstiness curves of deterministic traffic flows, namely that

$$b_{\sum_i A_i}(\sum_i c_i) \le \sum_i b_{A_i}(c_i), \tag{9}$$

where $\sum_i A_i$ denotes the traffic flow obtained by aggregating traffic flows $A_i$. This inequality follows straightforwardly from the definition (1) of $b_A(c)$.

In the following sections we find that the quality of the aggregate satisfies a certain optimality condition if the constituent flows are, in some sense, well-matched. Suppose the extremum in (8) is achieved at $(t^*, \theta^*)$. Assuming the requisite differentiability, (8) implies that

$$\frac{\partial I(c,b)}{\partial c} = t^*\theta^*, \qquad \frac{\partial I(c,b)}{\partial b} = \theta^*. \tag{10}$$

Thus $\theta^*$ characterizes the sensitivity of quality to changes in resources, while $t^*$ gives the relative sensitivity of the flows. Put another way,

$$t^* = \frac{\partial I(c,b)}{\partial c} \Big/ \frac{\partial I(c,b)}{\partial b} = -\frac{\partial b(c,\varepsilon)}{\partial c}\Big|_{I(c,b)=\varepsilon} \tag{11}$$

gives the (negative of the) slope of the burstiness curve corresponding to the quality $I(c,b)$. In view of the preceding paragraph, $t^*$ also determines the time over which arrivals build up to cause the queue to exceed a level $b$ per flow. Thus $t^*$ and $\theta^*$ together determine the manner in which the resource capacities $c$ and $b$ are exceeded. The flows are well-matched if the characteristic $t^*$ and $\theta^*$ are identical across flows. In a sense, the overflow-determining properties of the heterogeneous flows are rendered homogeneous. Since $t^* = -\frac{\partial b(c,\varepsilon)}{\partial c}$, an arbitrary set of flows can match at least their values of $t^*$ by choosing an appropriate point on the burstiness curve by which to specify their resource requirements $(c, b)$. The matching of the characteristic $t^*$ and $\theta^*$ is exploited in [4] in a different way. There, the goal is to construct a single substitution flow for a given set of (a large number of) flows, well-matched or not in our context, so that the aggregate and the single substitution flow are well-matched and have the same effective bandwidth $\mu(t,\theta)/t\theta$. They show that, under a condition that seems to hold in practice, this is achievable with fraction Brownian motion as the single substitution flow. This is applied to simplify traffic generation, and dynamic estimation of available capacity and buffer overflow probability.

The incentive for flows to match their $t^*$'s can be provided in the costs attached to the resources $b$ and $c$. These costs could be fixed, or per unit time. In Section 7 we show for a wide class of flows, that if the ratio of the cost of bandwidth to that of buffer space is $\gamma$, then individual users minimize their costs by choosing their point on the burstiness curve for which $t^* = \gamma$. The minimal cost can be specified directly in term of the MGF $\mu$ of the flow. This gives rise, in Section 8, to the notion of *minimal cost resource allocation*, in a network where the allocation of resources to users is limited not by the availability of resources, but rather by the willingness of users to pay for them. We show that delay constraints can be accommodated in this resource allocation scheme, and furthermore that the delay bounds are consistent with the SRSB and SRMB

| | |
|---|---|
| $A_{s,t}$ | arrival process |
| $A^n_{s,t}$ | $n$-fold aggregate of independent copies of $A_{s,t}$ |
| $b(c)$ | minimum buffer required to accommodate arrivals if served at rate $c$ |
| $D^n_{s,t}$ | departure process corresponding to $n$-fold aggregate arrivals |
| $\widetilde{D}^n_{s,t}$ | upper bound for departure process $D^n_{s,t}$ |
| $\varepsilon$ | asymptotic logarithmic tail probability |
| $H$ | Hurst parameter for fractional Brownian Motion |
| $I(c, b)$ | large deviation rate function for level $b$ of queue served at rate $c$ |
| $J(c, t, x)$ | upper bounding rate function for departure $D^n_{-t,0}$ |
| $\mu(t, \theta)$ | moment generating function of arrival process |
| $\mu^*(t, x)$ | Legendre transform of $\mu$ w.r.t. its second argument |
| $Q^n$ | queue length at time zero of $n$-fold aggregate arrival process |

Table 1: Table of notation used in the paper

rules for allocating resources to aggregations. In Section 9 we formulate cost based admission controls to aggregate channels. An aggregator of flows is able to pass on the economies of scale of multiplexing as reduced costs. We show how, if potential users of the aggregator knows their MGF they can tell whether the charge for a given quality reflects optimal use of aggregate resources, and whether the same quality at lower cost could be achieved without going through an aggregator. We conclude in Section 10 with some comments on the potential strengths of pricing based on resource allocations, and a comparison with some other proposed pricing methods. Finally, we collect together a list of the notation used in the paper in Table 1.

## 2 Shape function, Burstiness and Quality

The hypotheses form of (3) is the following. We detail the precise hypotheses in Appendix A.

**Theorem 1** *Under Hypothesis 1*

$$-I(c, b^+) \leq \liminf_{n\to\infty} n^{-1} \log \mathbf{P}[Q^n > nb] \leq \limsup_{n\to\infty} n^{-1} \log \mathbf{P}[Q^n > nb] \leq -I(c, b), \qquad (12)$$

*where $I(c, b^+)$ denotes the limit from the right in the second argument. A sufficient condition for the upper and lower bounds to coincide is that $\mu^*(t, \cdot)$ be continuous on $\mathbf{R}_+$.*

We do not reproduce the hypotheses in detail here, but will make some brief comments. For $\mu^{(n)}$ the MGF of $A^{(n)}$, the general aggregate flow described in the introduction, the main hypothesis is that the limit $\mu = n^{-1}\mu^{(n)}$ exists as $n \to \infty$ and is essentially smooth (see [25] for terminology). By the Gärtner-Ellis theorem (see [7]) this ensures that for each $t$, the pair $({A_t}^{(n)}, n)$ satisfies a *Large Deviation Principle* with rate function $\mu^*(t, \cdot)$. Then Theorem 1 says that, asymptotically as $n \to \infty$, the probability $\mathbf{P}[Q^n > nb]$ is equal to the least upper bound (7) as $t$ is varied, and that this bound is tight. This is the familiar large deviation heuristic that "rare events occur in the most likely way".

For each quality $\varepsilon > 0$ define the $\varepsilon$**–burstiness curve** (or indifference curve) $b(\cdot, \varepsilon)$ as the right-inverse of $I$ in its second component, i.e.,

$$b(c, \varepsilon) = \inf\{b > 0 \mid I(c, b) \geq \varepsilon\} \tag{13}$$

Thus $b(c, \varepsilon)$ is the minimum buffer allocation required for quality $\varepsilon$ at service rate (bandwidth) $c$. Let $a$ denote the mean arrival rate of a flow, i.e., $\mathbf{E}[A_t] = at$. We now find a variational expression for $b(\cdot, \cdot)$ in terms of the of the moment generating function $\mu$. For this we will need the following:

**Lemma 1** *$\mu^*(t, at) = 0$ and $x \mapsto \mu^*(t, x)$ is nondecreasing on $\mathbf{R}_+$ for $x \geq at$.*

**Proof of Lemma 1:** Since $\mu(t, 0) = 0$ then $\mu^*(t, at) \geq 0$. By Jensen's inequality, $\mu(t, \theta) \geq \theta \mathbf{E}[A_t] = \theta at$. Hence $\mu^*(t, at) = \sup_\theta (at\theta - \mu(t, \theta)) \leq 0$.

For $x > at$, $\mu(t, \theta) - \theta x \geq \theta(at - x) \geq 0$ if $\theta \leq 0$. Thus for $x > at$,

$$\mu^*(t, x) = \sup_{\theta \in \mathbf{R}} (x)\theta - \mu(t, \theta)) = \sup_{\theta \geq 0} (x\theta - \mu(t, \theta)) \tag{14}$$

which is nondecreasing in $x$. ■

By Lemma 1 we can define the inverse

$$\mu^*(t, \cdot)^{-1}(x) = \inf\{y > at \mid \mu^*(t, y) \geq x\}, \tag{15}$$

of $\mu^*(t, \cdot)$ on the domain $[at, \infty)$ on which it is non-decreasing. Now for each $t > 0$ set

$$b(c, \varepsilon, t) = \mu^*(t, \cdot)^{-1}(\varepsilon) - ct. \tag{16}$$

The next theorem gives the fundamental properties of $b_\varepsilon = b(\varepsilon, \cdot)$.

**Theorem 2** *(i) For each $\varepsilon > 0$, $b(c, \varepsilon)$ is a convex, nonincreasing function on $\mathbf{R}^+$.*

*(ii) $b \geq b(c, \varepsilon)$ iff $I(c, b) \geq \varepsilon$.*

*(iii) $b^* = b(c, \varepsilon) = \sup_{t>0} b(c, \varepsilon, t)$ is achieved at $t^*$ iff $\varepsilon = I(c, b^*) = \inf_{t>0} \mu^*(t, b^* + ct)$ is achieved at $t^*$.*

**Proof:** (i)

$$I(c, b) \geq \varepsilon \iff \forall_{t>0} : \mu^*(t, b + ct) \geq \varepsilon \iff \forall_{t>0} : b \geq b(c, \varepsilon, t) \iff b \geq \sup_{t \geq 0} b(c, \varepsilon, t). \tag{17}$$

Hence

$$b(c, \varepsilon) = \sup_{t>0} b(c, \varepsilon, t) \tag{18}$$

which, as a supremum of linear functions, is convex. Since for each $t > 0$, $b(c, \varepsilon, t)$ is decreasing in $c$, so is $b(c, \varepsilon)$.

(ii) This is a restatement of the definition (13).

(iii) We first prove the only if part of the statement. From the definition of $b(c, \varepsilon, t)$, then $b^* = \inf\{b > 0 \mid \mu^*(t^*, b + ct^*) \geq \varepsilon\}$ and so $\mu^*(t^*, b + ct^*) < \varepsilon$ for all $b < b^*$. Thus $\varepsilon \geq \liminf_{b \nearrow b^*} \mu^*(t^*, b + ct^*) \geq \mu^*(t^*, b^* + ct^*)$, where the second inequality follows since $\mu^*(t, \cdot)$ is, as a Legendre transform, lower semicontinuous. But from (ii), $b^* = b(c, \varepsilon)$ implies $I(c, b^*) \geq \varepsilon$. Thus

$$\varepsilon \leq I(c, b^*) = \inf_{t>0} \mu^*(t, b^* + ct) \leq \mu^*(t^*, b^* + ct^*) \leq \varepsilon, \tag{19}$$

and so the infimum is achieved at $t^*$.

For the if part, for any $\widetilde{b} \geq 0$

$$I(c, \widetilde{b}) = \mu^*(t^*, \widetilde{b} + ct^*) \implies \forall_{t>0} : \ \mu^*(t^*, \widetilde{b} + ct^*) \leq \mu^*(t, \widetilde{b} + ct) \tag{20}$$

$$\implies \forall_{t>0} : \ b(c, \varepsilon, t^*) \geq b(c, \varepsilon, t) \tag{21}$$

$$\implies b(c, \varepsilon) = b(c, \varepsilon, t^*). \tag{22}$$

The equality of $\widetilde{b}$ with $b^* = b(c, \varepsilon)$ then follows from the only if part. ■

In the next sections we shall give some examples of shape functions and burstiness curves for long and short-range dependent processes.

## 3 Departure processes and network quality

Here we show that, under technical conditions, quality propagates in a network in the sense that the burstiness of a traffic flow is not increased by passage through a FIFO queue.

We model the departure process by a fluid, so that arriving work is presented continuously to the output as it is processed. When the service rate is $nc$, the amount of work departing in the interval $[t, t')$ is $D^n_{t,t'} = \min[nc(t' - t), A^n_{t,t'} + Q^n_t - Q^n_{t'}]$ where $Q^n_t = \sup_{t'<t}(A^n_{t',t} - nc(t - t'))$ is the length of the queue of arrived work unprocessed at time $t$. For the present purposes it is both sufficient and simpler to work with the an upper bound for the departures:

$$\widetilde{D}^n_{t,t'} = cn(t' - t) + \min[0, \sup_{t''<t}(A^n_{t'',\, t'} - nc(t' - t'')] \geq D^n_{t,t'}. \tag{23}$$

Observe that $\widetilde{D}$ is obtained from $D$ upon replacing $Q^n_0$ by $0$.

**Theorem 3** *Assume the departure process $D^n_t$ satisfies the hypothesis of Theorem 1 for some* MGF*. Let $b, \widetilde{b}$ be the burstiness curves of arrivals and departures. Then for all $\varepsilon > 0$,* *(i)* $\widetilde{b}(\widetilde{c}, \varepsilon) \leq b(\widetilde{c}, \varepsilon)$ *for all* $\widetilde{c}$*.*

*(ii)* $\widetilde{b}(\widetilde{c}, \varepsilon) = 0$ *for all* $\widetilde{c} \geq c$*.*

We will establish a number of subsidiary results used in the proof of Theorem 3 and show how they combine to yield the result. After the proof we will illustrate the results with an example application for fractional Brownian motion arrivals. The next theorem characterizes the large deviation behavior of the departure process at the level of rate functions.

**Theorem 4** *Under the hypotheses of Theorem 1, for* $x \geq a$*,*

$$-J(c,t,x^+) \leq \liminf_{n\to\infty} n^{-1} \log \mathbf{P}[\widetilde{D}^n_{-t,\,0} \geq nx] \leq \limsup_{n\to\infty} n^{-1} \log \mathbf{P}[\widetilde{D}^n_{-t,\,0} \geq nx] \leq -J(c,t,x) \quad (24)$$

*and hence*

$$\limsup_{n\to\infty} n^{-1} \log \mathbf{P}[D^n_{-t,\,0} \geq nx] \leq -J(c,t,x) \quad (25)$$

*where*

$$J(c,t,x) = \begin{cases} \inf_{t'\geq t} \mu^*(t', x + c(t'-t)) & \text{if} \quad t \geq x/c \\ +\infty & \text{otherwise} \end{cases} \quad (26)$$

**Proof:** When $t < x/c$ then $\log \mathbf{P}[\widetilde{D}^n_{-t,\,0} \geq xn] = 0$, since then $\min[0, \sup_{t''\leq -t}(A^n_{t'',\,0} + nct'')] \leq 0 \leq n(x - ct) \geq 0$. So assume $t \geq x/c$. Then

$$\mathbf{P}[\widetilde{D}^n_{-t,\,0} \geq nx] = \mathbf{P}[\sup_{t''\leq -t} (A^n_{t',\,0} + nct') \geq n(x - ct)], \quad (27)$$

and the result follows essentially by the same argument as used in the proof of Theorem 1 since $D^n \leq \widetilde{D}^n$.
■

Now assume the departure process feeds a downstream queue whose service rate is $n\widetilde{c}$ for some $\widetilde{c} \in (a, c)$. The stationary queue length at time 0 is $\widetilde{Q}^n_0 = \sup_{t>0}(D^n_{-t,\,0} - n\widetilde{c}t)$.

**Theorem 5** *Assume the departure process* $\widetilde{D}^n_{-t,\,0}$ *satisfies the hypotheses of Theorem 1 for some* MGF*. Then*

$$\limsup_{n\to\infty} n^{-1} \log \mathbf{P}[\widetilde{Q}^n_0 > nb] \leq -\widetilde{I}(\widetilde{c}, b) \quad (28)$$

*where*

$$\widetilde{I}(\widetilde{c}, b) = \inf_{t'\geq b/(c-\widetilde{c})} \mu^*(t', b + \widetilde{c}t') \geq I(\widetilde{c}, b). \quad (29)$$

**Proof:** By Hypothesis 1 and the Gärtner-Ellis theorem, $(\widetilde{D}^n_{-t,\,0}, n)$ satisfies an LDP with a rate function which must, by Theorem 4, be equal to $J(c,t,\cdot)$. Hence by Theorem 1, $\mathbf{P}[\widetilde{Q}^n_t > nb]$ satisfies an upper bound of the form of that in (12) but with $I$ replaced by

$$\inf_{t>0} J(c,t,b+\widetilde{c}t) = \inf_{t\geq(b+\widetilde{c}t)/c}\ \inf_{t'\geq t} \mu^*(t', b+c(t'-t)+\widetilde{c}t) \tag{30}$$

$$= \inf_{t'\geq b/(c-\widetilde{c})}\ \inf_{t'\geq t\geq b/(c-\widetilde{c})} \mu^*(t', b+c(t'-t)+\widetilde{c}t) \tag{31}$$

$$= \inf_{t'\geq b/(c-\widetilde{c})} \mu^*(t', b+\widetilde{c}t'), \tag{32}$$

where the last equality follows since $x \mapsto \mu^*(t', x+\widetilde{c}t)$ is increasing by Lemma 1. ■

**Proof of Theorem 3:** (i) $b \geq b(\widetilde{c},\varepsilon) \Leftrightarrow I(\widetilde{c},b) \geq \varepsilon \Rightarrow \widetilde{I}(\widetilde{c},b) \geq \varepsilon \Leftrightarrow b \geq \widetilde{b}(\widetilde{c},\varepsilon)$. Hence $b(\widetilde{c},\varepsilon) \geq \widetilde{b}(\widetilde{c},\varepsilon)$.

(ii) Suppose $\widetilde{c} \geq c$. From (26), $J(c,t,b+\widetilde{c}t) = +\infty$ for all $b \geq 0$ and $t \geq 0$. Hence from (29), $\widetilde{I}(\widetilde{c},b) = +\infty$ for all $b \geq 0$. Then $b(\widetilde{c},\varepsilon) = 0$ by definition (13). ■

Note that the results for a single node can be composed along aggregation networks, i.e., networks in which each node combines multiple inputs into a single output.

**Example 1.** Consider the model of Norros [22] where $A_t = at + V_t$ where $V_t$ is standard fractional Brownian motion with Hurst parameter $H \in (1/2, 1)$. Since $V_t$ is Gaussian with zero mean and variance $t^{2H}$ we have

$$\mu(t,\theta) = at\theta + \theta^2 t^{2H}/2 \quad \text{and hence} \quad \mu^*(t,x) = \frac{(x-at)^2}{2t^{2H}}. \tag{33}$$

The variational calculation (8) gives

$$I(c,b) = \frac{1}{g(H)} b^{2-2H}(c-a)^{2H}, \qquad \text{where} \tag{34}$$

$$g(H) = 2H^{2H}(1-H)^{2-2H}. \tag{35}$$

Then it follows from (29) that

$$\widetilde{I}(\widetilde{c},b) = \begin{cases} 0 & & \text{if } \widetilde{c} \leq a \\ I(\widetilde{c},b) = & \frac{1}{2}\left(\frac{b}{1-H}\right)^{2-2H}\left(\frac{\widetilde{c}-a}{H}\right)^{2H} & \text{if } a < \widetilde{c} \leq c(1-H)+aH \\ \mu^*(b/(c-\widetilde{c}), bc/(c-\widetilde{c})) = & \frac{1}{2}\left(\frac{b}{c-\widetilde{c}}\right)^{2-2H}(c-a)^2 & \text{if } c(1-H)+aH \leq \widetilde{c} < c \\ +\infty & & \text{if } c \leq \widetilde{c}. \end{cases} \tag{36}$$

To see this note that for $\mu$ in (33), $t \mapsto \mu^*(t, b+ct)$ is convex and achieved its infimum at $t^* = Hb/((1-H)(c-a)$. The result then follows by comparing $t^*$ with $t_0 \equiv b/(c-\widetilde{c})$, the lower range of the infimum in (29). Now $t^* \geq t_0$ if $\widetilde{c} \leq c(1-H)+aH$, in which case $\widetilde{I}(\widetilde{c},b) = I(\widetilde{c},b)$. Otherwise, $\widetilde{I}(\widetilde{c},b)$ is the value of $\mu^*(t', b+\widetilde{c}t')$ at $t_0$. In Figure 1 we have plotted $I(\widetilde{c},b)$ and $\widetilde{I}(\widetilde{c},b)$ for $a=1, c=2$. The principle modification of $I$ to $\widetilde{I}$ is that $\widetilde{I}$ increases to $\infty$ as $\widetilde{c} \to c$. The fact that the arrival rate to the downstream

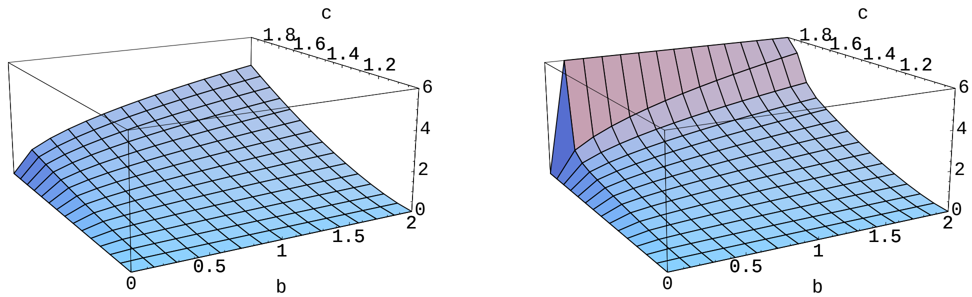


Figure 1: Non-decreasing property of shape function under passage through FIFO queue. Fractional Brownian motion input of Example 1. Left: inward shape function $I(c, b)$ for service rate $c$. Right: shape function $\widetilde{I}(\widetilde{c}, b)$ at downstream queue for service rate $\widetilde{c} \in (a, c)$, where $a$ is mean arrival rate.

queue is limited to $c$ means that large occupation of the downstream queue becomes increasingly unlikely as the downstream service rate $\widetilde{c}$ approaches $c$. The corresponding burstiness curves are

$$b(c, \varepsilon) = \begin{cases} \infty & \text{if } c \leq a \\ (1-H)(2\varepsilon(H/(c-a))^{2H})^{1/(2-2H)} & \text{if } c > a \end{cases} \tag{37}$$

for the input traffic flow. Downstream, the burstiness is

$$\widetilde{b}(\widetilde{c}, \varepsilon) = \begin{cases} b(\widetilde{c}, \varepsilon) & \text{if } \widetilde{c} \leq c(1-H) + aH \\ (2\varepsilon/(c-a)^2)^{1/(2-2H)}(c - \widetilde{c}) & \text{if } c(1-H) + aH \leq \widetilde{c} < c \\ 0 & \text{if } c \leq \widetilde{c} \end{cases} \tag{38}$$

We have plotted this for $\varepsilon = 4$ in Figure 2. This example will be used for illustration throughout the paper.

## 4 Resource allocation for aggregates

In this section and those following we develop criteria by which it can be determined whether aggregates of traffic flows, either homogeneous or heterogeneous, will enjoy a target quality. We use the shape function $I(c, b)$ for given service rate $c$ and buffer size $b$ as the measure of quality. We will generally work with a finite set of traffic flows characterized by their MGF $\mu_i$, where $i$ is an index running over the set of flows. Observe that $\sum_i \mu_i$ is the MGF of an aggregation of independent flows of MGF $\mu_i$.

Let $c_i$ and $b_i$ be the resources allocated to flow $i$. The motivation here is to examine the effect of resource combination rules for integrated service in which quality guarantees are only probabilistic in nature. Sometimes the target probability may not even be explicitly given. Our reference example is the proposed Controlled Load service specification [30]. For this, a flow is characterized by a Traffic Specification (TSpec) comprising leaky bucket parameters $(C, B)$, a rate $C$ and a bucket size $B$. (Peak rate, minimum policed unit and maximum packet size are also included, but these shall not concern us here). Packets conformant to a given TSpec should not experience queueing delay beyond that resulting from their own burstiness. Non-conformant packets may be sent by Best Effort service.

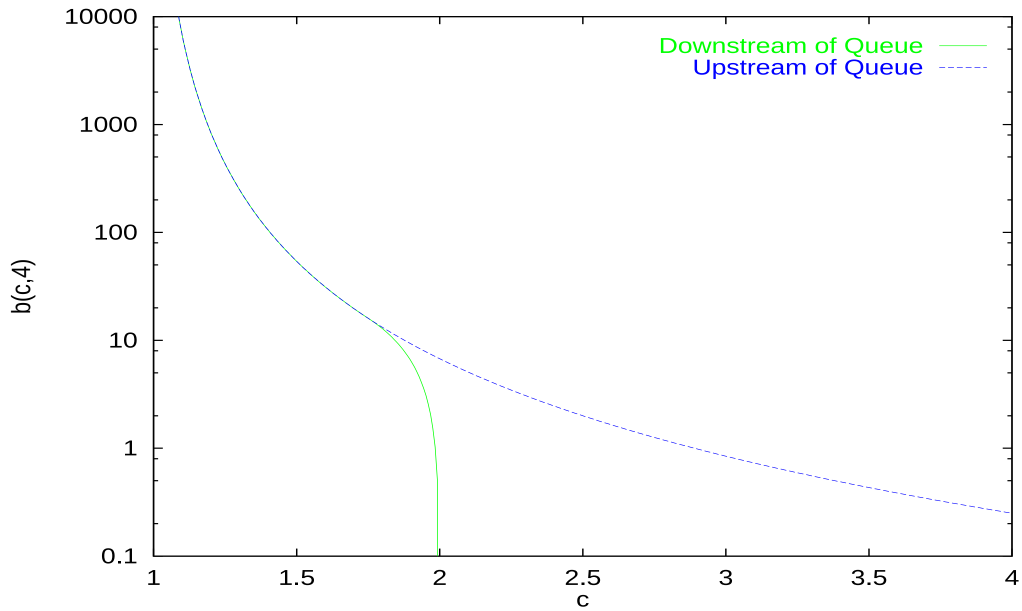


Figure 2: Non-increasing property of burstiness curve under passage through FIFO queue. Fractional Brownian motion input of Example 1. Burstiness upstream of queue, and downstream of queue.

Now, when traffic flows are to be combined at a network element, the summed TSpec of the combination (or more precisely, the $(C, B)$ part of the combined TSpec which we consider here) is to be derived from that of its constituents by the following rule:

**Sum rate, sum buffer (SRSB)** The aggregate flow is allocated bandwidth $c = \sum_i c_i$ and buffer $b = \sum_i b_i$.

The SRSB rule is conservative in finding the total resources to be allocated to flows shaped by leaky buckets with parameters $(c_i, b_i)$. However, statistical multiplexing is allowed provided that the use of shared resources by one flow does not unacceptably impact its use by other flows. Furthermore, per-node reshaping to the source TSpec leaky bucket is not part of the service specification. Hence, the use of shared resources is one way of accommodating a certain amount of non-conformance which might occur, for example, due to bursts queueing at upstream points. Thus non-conformance is to be expected in normal operation, although it should not occur too frequently. We may assign some target probability $\varepsilon$ to its occurrence. A specific property that we want to investigate is: if a set of traffic flows are non-conformant to their Tspec with some probability $\varepsilon$, can the same be said when their aggregate shares resources under the SRSB rule? This we investigate in Section 5 for sources which are characterized by an MGF.

Since we expect SRSB to be conservative, it is also interesting to ask whether resource allocation rules for aggregates can maintain quality while allocating less resources that SRSB. In Section 6 we investigate this question for homogeneous aggregates under the following rule:

**Sum rate, max buffer (SRMB)** The aggregate flow is allocated bandwidth $c = \sum_i c_i$ and buffer $b = \max_i b_i$.

## 5 Summed Rate Summed Buffer Allocations

For $n$-fold homogeneous aggregate under SRSB we can easily bound the quality of the aggregate in terms of the qualities of the constituent flows. When bandwidth and buffer allocation to individual flows are identical across flows then the quality of the aggregate is $n$ times that available to each component.

**Theorem 6** *For $n \in \mathbf{N}$,*

*(i)* $n \min_i I^\mu(c_i, b_i) \;\le\; I^{n\mu}(\sum_i c_i, \sum_i b_i) \;\le\; n \max_i I^\mu(c_i, b_i)$.

*(ii) When $c_i = c$ and $b_i = b$ for all $i$,*

$$I^{n\mu}(nc, nb) = nI^\mu(c, b), \quad \text{and hence} \quad b^{n\mu}(nc, n\varepsilon) = nb^\mu(c, \varepsilon). \tag{39}$$

**Proof:** (i) First note that $I^{n\mu}(\sum_i c_i, \sum_i b_i) \;=\; nI^\mu(\bar{c}, \bar{b})$ where $\bar{b} = \sum_i b_i/n$ and $\bar{c} = \sum_i c_i/n$ are average resources. Hence we only need to show

$$\min_i I^\mu(c_i, b_i) \;\le\; I^\mu(\bar{c}, \bar{b}) \;\le\; \max_i I^\mu(c_i, b_i). \tag{40}$$

For each $t > 0$ there exists $i(t)$ and $j(t)$ such that

$$b_{i(t)} + c_{i(t)}t \;\le\; \bar{b} + \bar{c}t \;\le\; b_{j(t)} + c_{j(t)}t$$

and hence by Lemma 1

$$\mu^*(t, b_{i(t)} + c_{i(t)}t) \;\le\; \mu^*(t, \bar{b} + \bar{c}t) \;\le\; \mu^*(t, b_{j(t)} + c_{j(t)}t).$$

Since this holds for all $t > 0$, taking infimum yields the result.

(ii) Clearly the above inequalities become equalities when $b_i$ and $c_i$ are identical for all $i$. Then the statement on the burstiness curves follows from Thm. 2(ii). ■

This result also illustrates the delicacy in aggregating flows and resources if quality is to be additive as in (39). For this, not only must the constituent flows be homogeneous, but each must also have been allocated the same resources $(c, b)$. Conversely, it is possible to allocate to a group of homogeneous flows different $(c_i, b_i)$, to achieve the same or different individual qualities $\varepsilon_i$, in such a way that the quality of the aggregate is strictly less than the sum available to components; see end of Example 2.

For heterogeneous aggregates, the properties of the shape function are somewhat more complicated. The following property of Legendre transforms will enable us to decompose the shape function of a aggregate in terms of that of is constituents; see Rockafellar [25], Theorem 16.4 and Corollary 26.3.2 for details.

**Lemma 2** *(i) Let $f_i$ be a finite set of proper convex functions on $\mathbf{R}$ for which the relative interiors of the effective domains have a point in common. Then*

$$(\sum_i f_i)^*(x) = \inf_{x_i : \sum_i x_i = x} \sum_i f_i^*(x_i), \tag{41}$$

*and the infimum is attained.*

*(ii) Moreover, if the $f_i$ are essentially smooth, so is $f = \sum_i f_i$, and the infimum is attained at $x_i = f_i'(\theta)$ where $\theta$ is the unique solution of $x = f'(\theta)$.*

Our first application of this result will be to show that quality is not impaired under SRSB. For the next two results $\mu = \sum \mu_i$ will denote the superposition of flows $i$ with MGF$\mu_i$, $I^\mu$ its shape function and $b^\mu$ its burstiness.

**Theorem 7** *(i) The quality of a heterogeneous aggregate under SRSB is at least as great as the minimum quality of its components, i.e.,*

$$I^\mu(\sum_i c_i, \sum_i b_i) \geq \min_i I^{\mu_i}(c_i, b_i). \tag{42}$$

*(ii) Hence $b^\mu(\sum_i c_i, \varepsilon) \leq \sum_i b^{\mu_i}(c_i, \varepsilon)$ for all $\varepsilon > 0$.*

**Proof:** (i) For simplicity assume at first that the $t$-infimum in (8) for $I^\mu(\sum_i c_i, \sum_i b_i)$ is attained. (If not, we can pick a $t$ for which the infimum is attained arbitrarily closely). Then $I^\mu(\sum_i c_i, \sum_i b_i) = \mu^*(t, \sum_i(b_i + c_i t)) = (\sum_i \mu_i)^*(t, \sum_i(b_i + c_i t)) = \sum_i \mu_i^*(t, x_i)$ with $x_i$ as in Lemma 2(i). For some $j$ we must have $x_j \geq b_j + c_j t$, otherwise we would have the contradiction that $\sum_i x_i < \sum_i(b_i + c_i t)$. Thus

$$I^\mu(\sum_i c_i, \sum_i b_i) \geq \mu_j^*(t, x_j) \geq \mu_j^*(t, b_j + c_j t) \geq \inf_t \mu_j^*(t, b_j + c_j t) = I^{\mu_j}(c_j, b_j) \geq \min_i I^{\mu_i}(c_i, b_i). \tag{43}$$

(ii) By (i) and Theorem 2(ii)

$$\forall_i b^{\mu_i}(c_i, \varepsilon) \leq b_i \Leftrightarrow \forall_i I^{\mu_i}(c_i, b_i) \geq \varepsilon \Rightarrow I^\mu(\sum_i c_i, \sum_i b_i) \geq \varepsilon \Leftrightarrow b^\mu(\sum_i c_i, \varepsilon) \leq \sum_i b_i. \tag{44}$$

■

Theorem 7(ii) can be viewed as a stochastic version of the aggregate inequality (9) for the deterministic burstiness curves (1). Recall that the deterministic burstiness of (1) and (9) shows that the burstiness curve of an aggregate, as defined in (1) to guarantee zero loss, is always smaller than the sum $\sum_i b_{A_i}(c_i)$ of individual burstiness curves. Here we have shown that the same holds true for $\varepsilon_i$-burstiness of the aggregate and its components. Thus we have achieved one of the aims set out in the previous section.

However, we expect to be able to do better than Theorem 7 and obtain more than the minimal quality of a constituent flow in an aggregate. As we shall see now, this can happen only when the traffic flows are, in some sense, well matched.

**Theorem 8** *Let the infimum in (8) for $I^\mu(\sum_i c_i, \sum_i b_i)$ be attained at $t^*$.*

*(i) Let the supremum in $\mu_i^*(t^*, b_i + c_i t^*)$ be attained at the same point $\theta$ for all $i$. Then*

$$I^\mu(\sum_i c_i, \sum_i b_i) \quad \geq \quad \sum_i I^{\mu_i}(c_i, b_i) \tag{45}$$

*with equality iff the infimum in $I^{\mu_i}(c_i, b_i)$ is attained at $t^*$ for all $i$. Hence $b^\mu(\sum_i c_i, \sum_i \varepsilon_i) \leq \sum_i b^{\mu_i}(c_i, \varepsilon_i)$ where $\varepsilon_i = I^{\mu_i}(c_i, b_i)$.*

*(ii) Let the infimum in (8) for $\varepsilon_i := I^{\mu_i}(c_i, b_i)$ be attained at the same point $\hat{t}$. Then*

$$I^\mu(\sum_i c_i, \sum_i b_i) \quad \leq \quad \sum_i I^{\mu_i}(c_i, b_i) \tag{46}$$

*with equality iff $\hat{t} = t^*$ and the supremum in $\mu_i^*(t^*, b_i + c_i t^*)$ is attained at the same $\theta$ $\forall$ $i$. Hence $b^\mu(\sum_i c_i, \sum_i \varepsilon_i) \geq \sum_i b^{\mu_i}(c_i, \varepsilon_i)$*

**Proof:** (i)

$$\begin{aligned} I^\mu(\sum_i c_i, \sum_i b_i) &= (\sum_i \mu_i)^*(t^*, \sum_i (b_i + c_i t^*)) && (47)\\ &= \sum_i \mu_i^*(t^*, b_i + c_i t^*) \qquad \text{by Lemma 2,} && (48)\\ &\geq \sum_i I^{\mu_i}(c_i, b_i), && (49)\end{aligned}$$

with equality iff the infimum for $I^{\mu_i}(c_i, b_i)$ is attained at $t^*$ also. (ii)

$$\begin{aligned} \sum_i I^{\mu_i}(c_i, b_i) &= \sum_i \mu_i^*(\hat{t}, b_i + c_i \hat{t}) && (50)\\ &\geq \mu^*(\hat{t}, \sum_i (b_i + c_i \hat{t})) \qquad \text{by Lemma 2,} && (51)\\ &\geq \mu^*(t^*, \sum_i (b_i + c_i t^*)) = I^\mu(\sum_i c_i, \sum_i b_i) && (52)\end{aligned}$$

with equality iff the supremum in $\mu_i^*(t^*, b_i + c_i t^*)$ is attained at the same $\theta$ for all $i$, and $\hat{t} = t^*$ ■

The importance of establishing the conditions in the foregoing theorems is that without them, the effect on quality of adding new components to an aggregate is uncontrolled, even under the SRSB rule. Indeed, the following example shows that it is possible to have $I(\sum_i c_i, \sum_i b_i) < \max_i I_i(c_i, b_i)$.

**Example 2.** Consider $n$ flows of fractional Brownian motion of Example 1: $A_i(t) = a_i t + V_i(t)$, $i = 1, \ldots, n$. From Example 1 the shape function of flow $i$ is:

$$I^{\mu_i}(c_i, b_i) = \frac{1}{g(H)}\, b_i^{2-2H}\, (c_i - a_i)^{2H}$$

where $g(H)$ is defined by (35). For the superposition $A(t) = \sum A_i(t)$ we have

$$\mu(t,\theta) = at\theta + \theta^2 T(t)/2 \quad \text{and hence} \quad \mu^*(t,x) = \frac{(x-at)^2}{2T(t)}$$

where $a = \sum a_i$ and $T(t) = \sum t^{2H_i}$. For the special case where the Hurst parameters $H_i = H$ are identical the shape function of the superposition is:

$$I^{\mu}(\sum c_i, \sum b_i) = \frac{1}{g(H)}\, \left(\frac{\sum b_i}{\sqrt{n}}\right)^{2-2H} \left(\frac{\sum (c_i - a_i)}{\sqrt{n}}\right)^{2H}$$

Hence the quality of the aggregate under SRSB is less than the quality of a constituent flow $j$, i.e.,

$$I^{\mu}(\sum c_i, \sum b_i) \quad < \quad I^{\mu_j}(c_j, b_j) \tag{53}$$

if and only if

$$\left(\frac{\sum b_i}{\sqrt{n}}\right)^{2-2H} \left(\frac{\sum (c_i - a_i)}{\sqrt{n}}\right)^{2H} \quad < \quad b_j^{2-2H}(c_j - a_j)^{2H}. \tag{54}$$

This is satisfied for $j = 2$ in the RHS of (54) when $n = 2$, $b_1 = c_1 - a_1 = 1$ and $b_2 = c_2 - a_2 = 3$ in which case:

$$\text{LHS } = 4^2/2 = 8 \quad < \quad 3^2 = \text{ RHS}.$$

Note that (53) can hold even when $a_1 = a_2$, i.e., the quality of the aggregate is strictly less than the maximum of individual quality even when the flows are homogeneous.

The previous results show that under the particular conditions that either the extremal $t$'s or $\theta$'s (or both) from (8) are equal, we can bound the quality of the aggregate by the sum of the qualities of its constituents. In subsequent sections we shall discuss the extent to which it is possible for flows to be well-matched by equality of their extremal $\theta$ or $t$. It is nevertheless desirable to obtain such bounds for the quality of the aggregate under some more general condition of similarity between the flows. We now establish such a result, by comparing the quality of the aggregate with that of a constituent flow that is 'relatively large' but is allocated a 'relatively small share' of resources, and with that of a constituent flow that is 'relatively small' but is allocated a 'relatively large share' of resources.

**Theorem 9** *For $n$ heterogeneous flows, if for a particular $i$, $l_i b_i \le \sum_j b_j$, $l_i c_i \le \sum_j c_j$ and $\sum_j \mu_j(t,\theta) \le l_i \mu_i(t,\theta)$ for all $t > 0$, $\theta > 0$, for some $l_i$, then*

$$l_i I^{\mu_i}(c_i, b_i) \;\le\; I^{\mu}(\sum_j c_j, \sum_j b_j). \tag{55}$$

*Similarly, if for a particular $k$, $\sum_j b_j \le u_k b_k$, $\sum_j c_j \le u_k c_k$ and $u_k \mu_k(t,\theta) \le \sum_j \mu_j(t,\theta)$ for all $t > 0$, $\theta > 0$, for some $u_k$, then*

$$I^{\mu}(\sum_j b_j, \sum_j c_j) \;\le\; u_k I^{\mu_k}(c_k, b_k). \tag{56}$$

**Proof of Theorem 9:** For the first inequality, we have

$$I^{\mu_i}(c_i, b_i) \quad \le \quad \inf_{t>0} \sup_{\theta>0} \left(\frac{1}{l_i}\sum_j (b_j + c_j t)\theta - \frac{1}{l_i}\sum_j \mu_j(t,\theta)\right) \quad = \quad \frac{1}{l_i} I^{\mu}(\sum_j c_j, \sum_j b_j). \tag{57}$$

The second inequality is proved in a similar way. ■

## 6 Summed Rate Maximum Buffer for Associated Flows

In the previous section we explored the extent to which increased quality could be obtained for an aggregate, as compared with its constituents, by SRSB allocation of the aggregate bandwidth and buffer $(\sum_i c_i, \sum_i b_i)$. Another approach to making use of the economies of scale is to offer less than the aggregate resources to the aggregate, but to have the aggregate quality no worse than that of the constituents. Theorem 7 showed that this is possible with SRSB allocation. In this section we show that this is possible for a wide class of traffic flows under SRMB allocation, i.e. of $(\sum_i c_i, \max_i b_i)$.

In part of what follows we shall consider traffic flows with the following correlation property. We say that an arrival process $A$ is **associated** if all its increments over disjoint time intervals are associated, i.e., if for all increasing function $f$ and $g$ we have that

$$\mathbf{E}[f(A_{t_1,t_2}) g(A_{t_3,t_4})] \ge \mathbf{E}[f(A_{t_1,t_2})]\mathbf{E}[g(A_{t_3,t_4})], \tag{58}$$

for disjoint time intervals $[t_1, t_2)$ and $[t_3, t_4)$. In particular, taking $f, g$ to be the identity we see that the increments are positively correlated when they are associated. Taking $f(x) = g(x) = e^{\theta x}$ we see that for associated traffic flows, $\mu(\cdot, \theta)$ is superadditive for $\theta \ge 0$, i.e.,

$$\mu(t+s, \theta) \ge \mu(t,\theta) + \mu(s,\theta), \quad \theta \ge 0. \tag{59}$$

A simple example of a process with associated increments is a two-level Markov process in which successive arrivals are positively correlated, see Section 4 of Botvich and Duffield [2]. But generally, positive correlations are not a sufficient condition for association. We remark that one could establish (59) empirically for a given traffic flow. The power of the association property is that it allows one to establish bounds for resource usage in terms of the corresponding behaviour as if the arrival process had independent increments, as we now see.

**Theorem 10** *Under SRMB, the quality available to a homogeneous aggregate is at least as great as that of each component. In other words: letting $\mu$ be the* MGF *of an associated traffic flow, and $n \in \mathbf{N}$, then*

$$I^{n\mu}(nc, b) \ge I^{\mu}(c, b), \quad \text{and hence} \quad b^{n\mu}(nc, \varepsilon) \le b^{\mu}(c, \varepsilon). \tag{60}$$

**Proof:** We first establish that for $n \in \mathbf{N}$ and $x > at$,

$$\mu^*(t,x) \leq n\mu^*(t/n, x/n). \tag{61}$$

From (i), $\mu(nt,\theta) \geq n\mu(t,\theta)$ for $\theta \geq 0$. As in Lemma 1, the supremum in $\mu^*(t,x) = \sup\{\theta x - \mu(t,\theta)\}$ can be restricted to positive $\theta$ when $x > at$. Hence, for such $x$, $\mu^*(nt,x) \leq (n\mu)^*(t,x) = n\mu^*(t,x/n)$.

Now, $I^{n\mu}(nc,b) = \inf_{t>0}(n\mu)^*(t,nct+b) = \inf_{t>0} n\mu^*(t,ct+b/n) \geq \inf_{t>0}\mu^*(tn,ctn+b) = \inf_{t>0}\mu^*(t,ct+b) = I^{\mu}(c,b)$, where the inequality follows from (61) and the last equality by rescaling the (dummy) variable $t$. Then the statement on the burstiness curves follows from Thm. 2(ii): $b \geq b^{\mu}(c,\varepsilon) \Leftrightarrow I^{\mu}(c,b) \geq \varepsilon \Rightarrow I^{n\mu}(nc,b) \geq \varepsilon \Leftrightarrow b \geq b^{n\mu}(nc,\varepsilon)$. ■

The assumption that a traffic flow has associated increments turns out to have strong ramifications for the form of the shape function $I(c,b)$, amongst which are that it is asymptotically linear in $b$. In particular, suppose there exists a strictly positive solution $\delta$ of the equation $\lambda(\delta) = c\delta$, where $\lambda$ is as in Hypothesis 1 in Appendix A. Suppose also that the limit $\nu = -\lim_{t\to\infty}\mu(t,\delta)$ exists. Then $\lim_{b\to\infty}(I(c,b) - \delta b) = \nu$; see Theorems 3 and 4 of [2]. Thus, the restriction to associated traffic flows would appear to rule out certain classes of traffic, in particular long range dependent arrival flows; the asymptotic behavior of $I$ is not in general linear for these flows, see Duffield [8, 9]. However, in order for Theorem 10 to hold, it is sufficient that (61) hold at the extremal $t$ for $I^{n\mu}$. Thus only the details of the arrival process at the governing time-scale need be relevant, not the whole process.

## 7 The relation of cost and quality

In this section we introduce cost structures for bandwidth $c$ and buffer $b$. Cost will act as a differentiator between points of the burstiness curve. A **cost structure** for a single set of resources $(c,b)$ is a pair $(\alpha,\beta) \in \mathbf{R}_+^2$. Their interpretation is that

$$w = \alpha c + \beta b \tag{62}$$

is the cost of making the allocation $(c,b)$.

Let $\mu$ be a MGF with $\varepsilon$–burstiness $b_\varepsilon$. Define the **$\varepsilon$–minimal cost** $M(\gamma,\varepsilon)$ for all $\gamma \geq 0$ by

$$M(\gamma,\varepsilon) = \inf_{c>0}(b(c,\varepsilon) + \gamma c). \tag{63}$$

For each elementary cost structure $(\alpha,\beta)$, $\beta M(\beta^{-1}\alpha,\varepsilon)$ is the minimal cost at which traffic flow with an $\varepsilon$–burstiness curve $b_\varepsilon$ can attain a quality $\varepsilon$.

At an informal level we can already see a striking relation between cost and quality. $b_\varepsilon$ is convex, and if it is also differentiable then from (63) we see that the minimal cost is attained at that $c$ for which $b'_\varepsilon(c) = -\gamma$. But as we saw by an informal argument in the introduction, (10–11) tell us that, with sufficient differentiability, $b'_\varepsilon(c) = -t^*$, where $t^*$ achieves the infimum in (8). The conclusion from this would then be

*For a given cost structure $(\alpha, \beta)$, a flow achieves minimal cost for a given quality $\varepsilon$ when its resources $(c, b)$ are such that $t^* = \beta^{-1}\alpha$. In this case the minimal cost is $\alpha c + \beta b$.*

We will now state more precisely sufficient conditions for the above to be true; differentiability of all functions concerned is not required. We call an inverse of $M(\gamma, \cdot)$, namely

$$K(\gamma, x) = \sup\{\varepsilon > 0 \mid M(\gamma, \varepsilon) \leq x\} \tag{64}$$

the $x$**–maximal quality**. For the elementary cost structure $(\alpha, \beta)$, $K(\alpha/\beta, x/\beta)$ is the maximal quality available at cost $x$.

The key to understanding the relation between cost and quality lies in identifying a convex duality between costs $M_\varepsilon$ and burstiness curves $b_\varepsilon$. Let $\mathcal{B}$ denote the set of closed convex nonincreasing extended real-valued functions on $\mathbf{R}_+$, and $\mathcal{M}$ denote the set of concave nondecreasing extended real-valued functions $M$ on $\mathbf{R}_+$ such that $-M$ is closed. There is a duality between these sets: with each element $b$ of $\mathcal{B}$ we can associate a unique element $M = \Phi b$ of $\mathcal{M}$ and likewise with each element $M$ of $\mathcal{M}$ an element $b = \Xi M$ of $\mathcal{B}$ where

$$M(\gamma) = (\Phi b)(\gamma) = \inf_{c>0}(b(c) + \gamma c), \qquad b(c) = (\Xi M)(c) = \sup_{\gamma>0}(M(\gamma) - \gamma c), \tag{65}$$

and $\Phi = \Xi^{-1}$ on $\mathcal{B}$ while $\Xi = \Phi^{-1}$ on $\mathcal{M}$. Both $\Phi$ and $\Xi$ extend to arbitrary real functions, but these extensions fail to be injective.

Let $\partial f$ denote the subgradient of a convex function $g$, i.e.

$$\partial f(x) = \{z \in \mathbf{R} \mid f(y) - f(x) \geq z(y - x)\}. \tag{66}$$

When $f$ is differentiable at $x$ then $\partial f(x) = \{f'(x)\}$. Let $(\partial f)^{-1}$ denote the inverse of $\partial f$ in the sense of multivalued mappings, i.e., $(\partial f)^{-1}(y) = \{x : y \in \partial f(x)\}$. For a concave function $g$, set $\delta g(x) = -(\partial(-g))(x)$.

By adapting some standard results from convex analysis to the present convex-concave duality (see e.g. Theorem 23.5 of [25]) one obtains the following useful characterization.

**Proposition 1** *Let $b \in \mathcal{B}$, $M \in \mathcal{M}$ with $M = \Phi b$ and $b = \Xi M$. Then the following are equivalent for $c, t > 0$:*

*(i)* $t \in -\partial b(c)$

*(ii)* $c't + b(c')$ *achieved its infimum in* $c'$ *at* $c$.

*(iii)* $c \in \delta M(t)$.

*(iv)* $M(t') - ct'$ *achieves its supremum in* $t'$ *at* $t$.

It will be convenient to use $\mu_\varepsilon^{*,-1}(t)$ as a shorthand for $\mu^*(t,\cdot)^{-1}(\varepsilon)$, and likewise $M_\varepsilon$ for $M(\cdot,\varepsilon)$.

**Theorem 11** *(i) The set $\mathcal{M}_\varepsilon$ of closed $\varepsilon$–minimal costs $M_\varepsilon$ and the set $\mathcal{B}_\varepsilon$ of closed $\varepsilon$–burstiness curves $b_\varepsilon$ is dual in the sense of (65).*

*(ii) For each $\varepsilon > 0$, $M(\cdot,\varepsilon) = \Phi b_\varepsilon = \Phi\Xi\mu_\varepsilon^{*,-1}$ is the concave hull of $\mu_\varepsilon^{*,-1}$. Hence*

$$M_\varepsilon(t) \geq \mu_\varepsilon^{*,-1}(t) \quad \text{for all} \quad t,\varepsilon > 0. \tag{67}$$

*(iii) For all $\gamma > 0$,*

$$\varepsilon \leq \mu^*(\gamma, M_\varepsilon(\gamma)) \quad \forall\varepsilon > 0 \quad \textit{and hence} \quad K(\gamma,x) \leq \mu^*(\gamma,x) \quad \forall x > 0. \tag{68}$$

**Proof:** (i) $b_\varepsilon$ is convex and $M_\varepsilon = \Phi b_\varepsilon$. Hence if $b_\varepsilon$ is also closed then $b_\varepsilon = \Xi M_\varepsilon = \Xi\Phi b_\varepsilon$. Similarly, $M_\varepsilon$ is concave, so when it is closed we have $M_\varepsilon = \Phi b_\varepsilon = \Phi\Xi M_\varepsilon$.

(ii) Since $b_\varepsilon = \Xi\mu_\varepsilon^{*,-1}$, $M_\varepsilon$ is the concave hull of $\mu_\varepsilon^{*,-1}$, from which (67) follows.

(iii) Applying $\mu^*(t,\cdot)$ to both sides of (67) with $t = \gamma$, then

$$\varepsilon = \mu^*(\gamma, \mu_\varepsilon^{*,-1}(\gamma)) \leq \mu^*(\gamma, M_\varepsilon(\gamma)). \tag{69}$$

Here, the equality follows from the lower semicontinuity of $\mu^*(t,\cdot)$, and the inequality since it is nondecreasing on $[at,\infty)$. The inequality for $K$ then follows from its definition. ■

**Theorem 12** *Fix $\gamma,\varepsilon > 0$ and suppose $b^* = b_\varepsilon(c) = \sup_{t>0} b_\varepsilon(c,t)$ is attained for all $c$. Then for all $\gamma > 0$,*

$$\varepsilon = \mu^*(t^*, M(\gamma,\varepsilon) + c^*(t^* - \gamma)) \tag{70}$$

*for some $c^* \in (\delta\Phi b_\varepsilon)(\gamma)$ and $t^* \in -\partial b_\varepsilon(c^*)$.*

**Proof:** (70) is clear at an informal level from the observation that since $M_\varepsilon$ is the concave hull of $\mu_\varepsilon^{*,-1}$, one should have $M_\varepsilon(\gamma) = \mu_\varepsilon^{*,-1}(t) + c(\gamma - t)$ for some $t \in \delta M_\varepsilon$. More precisely: by assumption, then for all $c$ there exists $t_c$ such that $b_\varepsilon(c) = b_\varepsilon(c,t_c)$. Thus by Theorem 2(iii),

$$\varepsilon = I(c, b_\varepsilon(c)) = \mu^*(t_c, b_\varepsilon(c) + ct_c). \tag{71}$$

By (67) $\mu_\varepsilon^{*,-1}(t_c) - ct_c \leq M_\varepsilon(t_c) - ct_c \leq b_\varepsilon(c)$. Hence $b_\varepsilon(c) = M_\varepsilon(t_c) - ct_c$, i.e., the supremum $\sup_{t>0}\{M_\varepsilon(t) - ct\}$ is attained at $t_c$, and by Proposition 1, $c \in \delta M_\varepsilon(t_c)$. Now take $c$ equal to some $c^* \in \delta M_\varepsilon(\gamma) = (\delta\Phi b_\varepsilon)(\gamma)$ and set $t^* = t_{c^*}$. Then by Proposition 1, $M_\varepsilon(\gamma) = b_\varepsilon(c^*) + \gamma c^*$, and combining with (71) yields (70). ■

Under several assumptions on either $b_\varepsilon$, $\mu_\varepsilon^{*,-1}$ or $M_\varepsilon$, we can show that $t^* = \gamma$ in (70).

**Theorem 13** *For any $\gamma > 0$ and $x > 0$*

$$\varepsilon = \mu^*(\gamma, M_\varepsilon(\gamma)), \qquad \text{and} \qquad K(\gamma, x) = \mu^*(\gamma, x) \tag{72}$$

*provided any one of the following is true:*

*(i)* $b_\varepsilon(c^*) = \sup_{t>0}(M_\varepsilon(t) - c^*t)$ *is uniquely attained.*

*(ii)* $b_\varepsilon$ *is differentiable*

*(iii)* $M_\varepsilon$ *is strictly concave;*

*(iv)* $\mu_\varepsilon^{*,-1}$ *is concave;*

*(v) for all* $b, c > 0$, $I(c, b) = \inf_t \mu^*(t, b + ct)$ *is uniquely attained.*

**Proof:** (i) By Proposition 1 both $t^*$ and $\gamma$ lie in $-\partial b_\varepsilon(c^*)$. If the supremum is uniquely attained then by the same proposition $\partial b_\varepsilon$ has a unique element. Hence $t^* = \gamma$ and (72) holds. The equality for $K$ then follows from its definition.

(ii,iii) $\partial b_\varepsilon$ is single valued $\Leftarrow$ $b_\varepsilon$ is differentiable $\Leftarrow$ $b_\varepsilon$ is essentially smooth $\Leftrightarrow$ $-M_\varepsilon$ is strictly convex (see Theorem 26.3 of [25]) $\Leftrightarrow$ $M_\varepsilon$ is strictly concave. The equality for $K$ then follows from its definition.

(iv) If $\mu_\varepsilon^{*,-1}$ is concave, then $\mu_\varepsilon^{*,-1} = M_\varepsilon$, from which (72) follows. The equality for $K$ then follows from its definition.

(v) $\varepsilon = I(c, b) = \inf_{t>0} \mu^*(t, b + ct)$ is uniquely attained $\Leftrightarrow$ $b = \sup_{t>0}(\mu_\varepsilon^{*,-1}(t) - ct)$ is uniquely attained (by Theorem 2) $\Rightarrow$ $b = \sup_{t>0}(M_\varepsilon(t) - ct)$ is uniquely attained; this follows by an argument similar to that in Theorem 12. Hence (72) follows by applying (i). The equality for $K$ then follows from its definition. ■

**Remarks:** The conditions (ii,iii) and condition (iv) address two ways in which $t^*$ in (70) may not equal $\gamma$. If $\mu_\varepsilon^{*,-t}$ is not concave then $\sup_{t>0}(\mu_\varepsilon^{*,-1}(t) - tc^*)$ could be attained at (say) two points $t_1, t_2$ with $t_1 < \alpha < t_2$, but at no points $t \in (t_1, t_2)$. However, the concave hull $M_\varepsilon$ of $\mu_\varepsilon^{*,-1}$ will be affine on $[t_1, t_2]$ (and hence not *strictly* concave), so that $\sup_{t>0}(M_\varepsilon(t) - tc^*)$ is attained at all $t \in [t_1, t_2]$, including $t = \alpha$. But if $\mu_\varepsilon^{*,-1}$ is concave to start with then it equals $M_\varepsilon$, and so $\sup_{t>0}(\mu_\varepsilon^{*,-1}(t) - tc^*)$ is attained at $t = \alpha$, albeit not necessarily uniquely. Finally, we remark that a sufficient condition for the infimum in (v) to be uniquely attained is that $\mu^*$ be (jointly) convex.

## 8 Resource allocation for minimal cost

If we assume that resources are available as required with charges according to the cost structure $(\alpha, \beta)$, then in order to allocate resources to a single flow, we must use the following **minimal cost resource allocation**:

(a) *Determine the* MGF $\mu$*; either by modelling, or measurement;*

(b) *Solve* $b'_\varepsilon(c) = \alpha/\beta$ *to find resources* $c^*$ *and* $b^* = b_\varepsilon(c^*)$ *which realize this quality. Allocate* $(c^*, b^*)$.

Note that the minimal cost $w = \alpha c^* + \beta b^*$ can be determined by Theorem 13 directly without having to find the resources $(c^*, b^*)$ in (b) above. We have not addressed here how to measure the MGF $\mu$: an approach to this are given in the work of McGurk and Walsh [20].

For the rest of this paper, by 'cost' we may mean $\alpha c + \beta b$, $\alpha c/\beta + b$, $M_\varepsilon(\alpha/\beta)$, or $\beta M_\varepsilon(\alpha/\beta)$; the meaning should be clear from the context.

**Example 3.** Consider the fractional Brownian Motion of Example 1. In a cost structure $(\alpha, \beta)$, then to find the resource allocation for minimal cost at quality $\varepsilon$ we find the solution $c^*$ of the equation $b'(c, \varepsilon) = -\alpha/\beta^{-1}$ with $b(c, \varepsilon)$ as in (37). A simple computation yields

$$c^* = a + \sqrt{2\varepsilon} H(\alpha/\beta)^{H-1} \quad \text{and} \quad b^* = b(c^*, \varepsilon) = \sqrt{2\varepsilon}(1 - H)(\alpha/\beta)^H. \tag{73}$$

One can confirm that the (normalized) cost $w = b^* + c^*\alpha/\beta$ satisfies $\mu^*(\alpha/\beta, w) = \varepsilon$.

If we assume that all resource allocation is done at minimal cost within the same cost structure $(\alpha, \beta)$, then it follows that all extremizing $t^*$ in (8) is $\alpha\beta^{-1}$ for all traffic flows, e.g., for all flows in an aggregate. Thus we obtain automatically the versions of previous theorems on quality for aggregates which presupposed equality of the various $t^*$.

**Theorem 14** *Let* $\mu = \sum_i \mu_i$. *Under any of the conditions (i) to (v) of Theorem 13*

*(i) Maximal quality is subadditive and minimal cost superadditive over aggregates, i.e,*

$$K_\mu(\gamma, \sum_i x_i) \leq \sum_i K_{\mu_i}(\gamma, x_i) \quad \text{and} \quad M_\mu(\gamma, \sum_i \varepsilon_i) \geq \sum_i M_{\mu_i}(\gamma, \varepsilon_i). \tag{74}$$

*Equality occurs when*

$${\mu_i}^{*\prime}(\gamma, x_i) = \mu^{*\prime}(\gamma, \sum_i x_i), \tag{75}$$

*i.e., the values of* $\theta^*$ *are identical across each* $\mu_i$ *and their sum* $\mu$. *Here* $\mu^{*\prime}$ *denotes the derivative of* $\mu^*$ *w.r.t. its second argument.*

*(ii) Lower bound: if all elements in an aggregate have individual quality at least* $\varepsilon$ *at optimal cost, then* $\varepsilon$ *is a lower bound for the quality of the aggregate at the same total optimal cost, i.e.,*

$$K_{\mu_i}(\gamma, y_i) \geq \varepsilon, \ \forall_i \quad \Longrightarrow \quad K_\mu(\gamma, \sum_i y_i) \geq \varepsilon. \tag{76}$$

**Example 4.** Consider $n$ groups of $m$ flows, all flows being independent, and the flows in group $i$ having common MGF $\mu_i$. Suppose each group is to be aggregated independently at quality $\varepsilon$. From Theorem 14(i) it follows that the total minimal cost of this arrangement is not less than that for any other grouping of the

$n \times m$ flows into $n$ groups. For consider a general such grouping as $\nu_k = \sum_{j=1}^m \mu_{i_{k,j}}$ for $k = 1, \ldots, n$. Then

$$\sum_{k=1}^n M_{\nu_k}(\gamma, \varepsilon) \geq \sum_{k=1}^n \sum_{j=1}^m M_{\mu_{i_{k,j}}}(\gamma, \varepsilon/m) = \sum_{k=1}^n m M_{\mu_k}(\gamma, \varepsilon/m) = \sum_{k=1}^n M_{m\mu_k}(\gamma, \varepsilon), \tag{77}$$

where the last equality follows from (72) by a scaling argument similar to that in Theorem 6.

We saw in the introduction that the optimizing $\theta$ in (8) can be interpreted as a quality sensitivity to resource. Cost structures allow us to interpret it as the sensitivity of maximal quality to optimal cost, since

$$\frac{\partial K(\gamma, x)}{\partial x} = \mu^{*\prime}(\gamma, x) = \theta^*. \tag{78}$$

Theorems 8(i) and 14(i) suggest that it might be possible to choose $(c_i, b_i)$ such that the quality $I^\mu(\sum_i c_i, \sum_i b_i)$ of the aggregate is at least $\sum_i I^{\mu_i}(c_i, b_i)$. To satisfy the sufficient condition in the theorems, however, would require coordinating the choice of $(c_i, b_i)$ for all flows such that the supremum in $\mu_i^*(t^*, b_i + c_i t^*)$ is attained at the same $\theta$ for all $i$. Can a cost structure provide the incentive for this, i.e., does there exists an elementary cost structure $(\alpha, \beta)$ such that, for *all* flows $i$, the minimizers $(c_i, b_i)$ of resource costs

$$w_i(c_i, b_i) \quad = \quad \alpha c_i + \beta b_i$$

achieve given qualities $\varepsilon_i$, i.e., $I_i(c_i, b_i) = \varepsilon_i$, and satisfy the condition of Theorem 8(i)? In fact, such a cost structure does not unfortunately exist in general, as the next example shows. The reason is that quality requirement that $I_i(c_i, b_i) = \varepsilon_i$ imposes an $\varepsilon_i$-burstiness curve $b_i(c)$ on which $(c_i, b_i)$ must lie. Minimizing the resource costs $w_i(c_i, b_i)$ over a specified $\varepsilon_i$-burstiness curve yields allocations $(c_i^*, b_i^*)$, which may not satisfy the condition of Theorem 8(i).

**Example 5.** Consider the fractional Brown Motion of Example 2 with:

$$I^{\mu_i}(c_i, b_i) = \frac{1}{g(H)}\, b_i^{2-2H}\, (c_i - a_i)^{2H}$$

and extremal $t_i$ and $\theta_i = \theta_i(t_i)$:

$$\begin{aligned} t_i \quad &= \quad \frac{H}{1-H} \frac{b_i}{c_i - a_i} \\ \theta_i \quad &= \quad \frac{2(1-H)}{g(H)} \frac{(c_i - a_i)^{2H}}{b_i^{2H-1}}. \end{aligned} \tag{79}$$

Given desired individual qualities $\varepsilon_i$ the $\varepsilon_i$-burstiness curve $b_i(c_i) = b_i(c_i, \varepsilon_i)$ is:

$$b_i(c_i) \quad = \quad (g(H)\varepsilon_i)^{\frac{1}{2-2H}}\, (c_i - a_i)^{-\frac{H}{1-H}} \tag{80}$$

To satisfy the sufficient condition in Theorem 8(i) we must have $\theta_i = \theta^*$ for all $i$. To be efficient we must have $b_i = b_i(c_i)$. Applying (79–80) to these two conditions yield the allocation $(c_i^*, b_i^*)$:

$$b_i^* \quad = \quad \frac{2(1-H)}{\theta^*} \varepsilon_i, \qquad c_i^* \quad = \quad a_i \;+\; (g(H)\varepsilon_i)^{\frac{1}{2H}}\, b_i^{\frac{H-1}{H}}$$

that achieves $\varepsilon_i$ and ensures $I(\sum_i c_i, \sum_i b_i) \geq \sum_i I^{\mu_i}(c_i, b_i)$. If $(c_i^*, b_i^*)$ also minimizes $w_i(c_i, b_i)$ then

$$b_i'(c_i^*) \;=\; -\frac{H}{1-H}(g(H)\varepsilon_i)^{-\frac{1}{2H}}(b_i^*)^{\frac{1}{H}} \;=\; \frac{\alpha}{\beta}. \tag{81}$$

Substituting $b_i^*$, we see that (81) holds if $\varepsilon_i$ are equal for all $i$, which may not be the case.

In Example 5 the quality of the aggregate will indeed be at least sum of the individual qualities at least cost if the constituent flows, without aggregation, desire the same (appropriate) quality $\varepsilon$. However this condition is generally not sufficient. In the next example the required condition is that a combination of desired quality and traffic characteristic be well matched (interestingly, the more random a flow the higher quality it must desire in order for the quality of the aggregate to be at least as the sum of individual qualities at least cost).

**Example 6.** Consider Gaussian flows: $A_i(t) = \lambda_i t + \sigma_i \sqrt{t} Z_i(t)$, $i = 1, \ldots, n$, where $Z_i(t)$ are white Gaussian. The shape function $I_i(c_i, b_i)$ and its minimizing $t_i$ and maximizing $\theta_i$ are

$$I^{\mu_i}(c_i, b_i) = \frac{2b_i(c_i - \lambda_i)}{\sigma_i^2}, \qquad t_i = \frac{b_i}{c_i - \lambda_i}, \qquad \theta_i = \frac{2(c_i - \lambda_i)}{\sigma_i^2}. \tag{82}$$

The $\varepsilon_i$-burstiness curve of flow $i$ is

$$b_i(c) \;=\; \frac{\varepsilon_i \sigma_i^2}{2(c_i - \lambda_i)}.$$

To satisfy the sufficient condition in Theorem 8(i) we must have $\theta_i = \theta^*$ for all $i$. Hence the allocation

$$c_i^* \;=\; \lambda_i + \frac{1}{2}\theta^* \sigma_i^2, \qquad b_i^* \;=\; b_i(c_i^*) \;=\; \frac{\varepsilon_i}{\theta^*}$$

achieves $\varepsilon_i$ and $I(\sum_i c_i, \sum_i b_i) \geq \sum_i I^{\mu_i}(c_i, b_i)$. If $(c_i^*, b_i^*)$ also minimizes $w_i(c_i, b_i)$ then

$$b_i'(c_i^*) \;=\; -\frac{2\varepsilon_i}{\theta^2 \sigma_i^2} \;=\; \frac{\alpha}{\beta}$$

i.e., $\varepsilon_i/\sigma_i^2$ must be equal for all $i$, which may not be the case.

**Resource allocation under delay constraints.** The cost-based resource allocation mechanism described above can be extended to cover delay sensitive flows. Under a delay constraint $T$, we required to allocate resources $(c, b)$ such that the buffer drain time $b/c < T$. Thus we must replace step (b) in the above rules for minimal cost allocation by the following.

(b$'$) *Solve* $b_\varepsilon'(c) = \alpha/\beta$ *to find resources* $(c^*, b^*)$ *which realize the quality* $\varepsilon$. *Solve* $b_\varepsilon(c) = cT$ *to find resources* $c_*$ *and* $b^* = b_\varepsilon(c_*)$ *which satisfy the delay constraint. Allocate resources* $(\max\{c^*, c_*\}, \min\{b^*, b_*\})$.

This extended resource allocation rule is also preserved by aggregations which use the SRSB and SRMB rules of Section 4 for allocation resource to aggregates. For if $b_i \leq c_i T$ for all $i$, then we have

$$\frac{\sum_i b_i}{\sum_i c_i} \leq T, \quad \text{and} \quad \frac{\max_i b_i}{\sum_i c_i} \leq T. \tag{83}$$

**Example 7.** Consider the fractional Brownian motion of Example 1, but now under a delay constraint $T$. If $(c^*, b^*)$ from (73) satisfy $b^*/c^* < T$ then we can use the resource allocation $(c^*, b^*)$. Otherwise we must use $(c_*, b_*)$ where $c_*$ is the root of the equation $cT = (1-H)(2\varepsilon(H/(c-a))^{2H})^{1/(2-2H)}$ and $b_* = b_\varepsilon(c_*)$.

## 9 Admission control, costs and incentives

We saw in the previous section that a single channel cannot in general provide additive quality across all potential flows. But we now show that when there are multiple channels with possibly different cost structures, it is possible to match flows to channels in order to promote matching of flow characteristics. Moreover, the cost structure provides a mechanism to encourage this which is tied closely to admission control. In what follows we assume that resources, buffer $b$ and bandwidth $c$ are available as required, subject to the charges for their use as specified by a cost structure.

We define **extended cost structure** as a triple $(\alpha, \beta, \theta) \in \mathbf{R}_+^3$. The role of $\alpha$ and $\beta$ is as in the cost structure defined previously. As the notation suggests, $\theta$ will play the role of the second argument of $\mu$. By specifying an extended cost structure we are, in effect, announcing the extremal variables $t = \alpha/\beta$ and $\theta$ in (8) at which optimal quality will be obtained. The following theorem is an easy consequence of Theorem 13 and the definition of the Legendre transform.

**Theorem 15** *Consider a channel with extended cost structure $(\alpha, \beta, \theta)$, and set $\gamma = \alpha\beta^{-1}$. A flow with* MGF *$\mu$ which satisfies any of the conditions of Theorem 13 can obtain quality $\varepsilon$ at cost $x$ if*

$$\mu(\gamma, \theta) \leq x\theta - \varepsilon. \tag{84}$$

*Moreover, the cost is minimal if $\mu'(\gamma, \theta) = x$ where $\mu'$ denotes the derivative of $\mu$ w.r.t. its second argument.*

When $\mu = \sum_i \mu_i$ is an aggregation of multiple flows, then we obtain from (84) the following **Cost Based Admission Control Rule** for admitting flows to use resources at a total fixed cost. Under the assumptions of Theorem 15:

> *Flows with* MGF *$\mu_i$ can obtain joint quality $\varepsilon$ at total cost x if $\sum_i \mu_i(\gamma, \theta) \leq x\theta - \varepsilon$.*

$(\gamma\theta)^{-1}\mu_i(\gamma, \theta)$ is familiar as an effective bandwidth as propounded by Kelly [14]. By choosing a value of $\theta$ we obtain an admission control rule which is additive in the flows through their effective bandwidths, a desirable property. In some sense the value of $\theta$ is arbitrary; choosing one value or another is simply specifying a condition for admission control, although this does not exclude the possibility that one choice of $\theta$ may lead to more efficient use of resources than another. This aspect is familiar, for the same reason, from the admission control schemes of Gibbens and Kelly [12] based on a simple characterization of the one-time marginal distribution of the arrival process.

However, the arbitrariness can be resolved somewhat in the presence of different channels, with different extended cost structures. If $\gamma, \theta$ and the actual charge $y_i$ to flows $\mu_i$ are known, then it can be determined to what extent $y_i$ differs from that required for quality to be maximal, i.e., $\mu_i'(\gamma, \theta)$. This provides the (owner of a) flow with a cost criterion by which to evaluate the service provided by different extended cost structures. With sufficiently many extended cost structures available, a flow could choose one for which the charge is closest to the optimal cost. Such behavior will lead to flows on individual channels becoming matched, in the sense that they share equal or closer values of $\theta$, and consequently their aggregate quality being maximized for the given total cost.

**Economies of scale.** It can also be determined to what extent a flow is benefiting from the potential economies of scale of multiplexing. Let us suppose for example, that a given channel is available in a "wholesale" and "retail" versions. These are distinct channels with extended cost structures $(\alpha, \beta, \theta)$ and $(\eta\alpha, \eta\beta, \theta)$ respectively, with $\eta > 1$. The channels have the same cost ratio $\gamma = \alpha/\beta$ and sensitivity $\theta$, but the resources are a factor $\eta$ more expensive for retail channel.

Suppose a flow is quoted a cost $x$ to gain admittance to the wholesale channel at aggregate quality $\varepsilon$. But according to Theorem 13, the retail channel could give quality $\varepsilon$ at cost $\eta\mu^*(\gamma, \cdot)^{-1}(\varepsilon)$. If this is less than $x$ then the optimal cost of the retail channel is less for the same quality.

**Example 8.** Consider the flow of Example 1: a flow $\mu$ with arrival process $A_t = at + V_t$ with $V_t$ fractional Brownian motion. One can check from (37) that this flow satisfies (ii) of Theorem 13. Suppose the target quality $\varepsilon$ is offered at the wholesale channel at cost $y$. Using (33) then we can tell that quality is not maximal at that cost if

$$y \geq \mu'(\gamma, \theta) = a\gamma + \theta\gamma^{2H}. \tag{85}$$

However, the retail channel will not be cheaper unless

$$y \geq \eta\mu^*(\gamma, \cdot)^{-1}(\varepsilon) = \eta(a\gamma + \sqrt{2\varepsilon\gamma^{2H}}). \tag{86}$$

Note that the inference on optimality is drawn without explicit reference to the other flows in the wholesale channel.

## 10 Conclusion: on the use of resource based pricing.

It has long been recognized that a flow that is carried lowers the value of the network for every other flow, and that usage-based pricing is a way to internalize this externality, aligning social and individual optimality, see e.g., MacKie-Mason and Varian [19]. Usage-based prices may have components such as an access charge, a connection charge per unit time, a volume charge, and a resource charge. Inclusion of these components generally have implications on the equilibrium utilization and social benefit and should be decided on social and economic grounds as well as traffic engineering ones. Nonetheless pricing (or some of its components) is a useful tool to induce more efficient sharing of network resources. For instance, based on effective bandwidth, Kelly [15] proposes a charging scheme with three components that encourages users to accurately estimate their mean rate, thus allowing more effective sharing of bandwidth. Distributed iterative schemes for setting usage-based prices in order to optimize resource allocation have been proposed, e.g., by Low and Varaiya [17], Murphy, Murphy and Posner [21] and de Veciana and Baldick [28].

The cost structure used in this paper, and also that in [17], is resource based where the price for a flow depends only on the resources buffer $b$ and bandwidth $c$ which are allocated, although the use of fixed and/or per unit time costs for these resources is envisaged. This has three useful consequences.

First, although the resource requirements via the MGF are quite sophisticated, the form of the cost itself is not, being just a linear combination of resources. Such costs are readily understandable.

Second, we have seen in Section 4 that $b$ and $c$ form part of the TSpec used by Controlled Load. Consequently, once the cost structure is determined, the prices for a given resource allocation under these emerging protocols are readily available. By contrast, the means by which QoS requirements are translated into resource allocations, such as by our use of MGF's, can be specific to an implementation, and need be performed only at a lower layer in the network protocol stack. Implementations might measure $\mu$.

Third since bandwidth and buffer are separately priced, their prices can convey to the traffic flows the relative availability of bandwidth and buffer. This allows individual flows to pick bandwidth and buffer allocations, based on their burstiness curves, that incurs the least cost. This was done in Low and Varaiya [17] for a network of deterministic flows where statistical multiplexing was excluded. When flows can be aggregated we have seen in Sections 5 and 6 that the quality of the aggregate is sensitive to well-matchedness of the constituent flows in terms of both their traffic characteristics and desired qualities. Here the resource costs, through their ratio, play the additional role of encouraging well-matchedness among flows. The consequent increase in quality is more easily achieved when different extended cost structures are available. For individual flows, the explicit relation between (optimal) cost and quality guides the choice of leaky bucket parameters used to shape the flow in order to pay the lowest cost for the desired quality.

## A Appendix: Hypotheses for Theorem 1

We state more the assumptions of Theorem 1. For each $n$ let $A^n$ be an arrival process (though not necessarily an aggregate of $n$ i.i.d. flows we deal with in the introduction). Let time be indexed by the set $T = \mathbf{R}_+$ or $\mathbf{Z}_+$. A *scaling function* is an increasing function from $T$ onto an unbounded subset of $\mathbf{R}_+$. For some scaling function $v$ define for each $n, t$ the MGF $\lambda^n(t, \cdot)$ by

$$\lambda^n(t,\theta) = \frac{1}{nv(t)} \log \mathbf{E}[e^{\theta v(t) A_t^n / t}] \tag{87}$$

for each $\theta \in \mathbf{R}$. In order to establish (3) for the asymptotic behavior as $n \to \infty$ of the queue length $Q^n$ with service rate $cn$ the following hypotheses are used:

**Hypothesis 1** *(i) The limits*

$$\lambda(t,\theta) = \lim_{n\to\infty} \lambda^n(t,\theta) \quad \text{and} \quad \lambda(\theta) = \lim_{t\to\infty} \lambda(t,\theta) \tag{88}$$

*exist as extended real numbers, the first limit existing uniformly for sufficiently large $t$.*

*(ii) Each $\lambda(t,\cdot)$ is essentially smooth.*

*(iii) There exists $\theta > 0$ such that $\lambda(t,\theta) - c\theta < 0$ for all $t$ sufficiently large.*

*(iv) For all $\varepsilon < 0 < \xi$,*

$$\lim_{m\to\infty} \limsup_{n\to\infty} n^{-1} \log \sum_{m' \geq m} e^{\varepsilon n v(m'\xi)} = -\infty. \tag{89}$$

*(v) ($T = \mathbf{R}_+$) For all $t \geq r \geq 0$ define $\widetilde{A}^n_{t,r} = \sup_{0<r'<r} A^n_{t-r'} - A^n_t$. Then for all $\theta \geq 0$*

$$\limsup_{r\to 0} \limsup_{n\to\infty} n^{-1} \sup_{t \geq 1/r} \log \mathbf{E}[e^{\theta v(t) \widetilde{A}^n_{t,r}/t}] \leq 0. \tag{90}$$

We remark that slightly stronger conditions were stated in Hypothesis 1 of [8]. (v) is automatically satisfied if the increments of $A$ are non-negative. Although the limit $\lambda(\theta)$ in (i) plays no direct role in the variational principle, its existence is required in order to establish the upper bound in Theorem 1. (iii) is a stability condition which is satisfied if the service rate $c$ is greater than the mean arrival rate $a$. (iv) is essentially technical.

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